

# A FEW OBSERVATIONS ON WEAVER'S QUANTUM RELATIONS

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**ABSTRACT.** In [Wea12], Weaver introduced the concept of quantum relation  $\mathcal{R}$  over a von Neumann algebra  $\mathcal{M}$ . When  $\mathcal{M}$  is either finite dimensional or discrete and abelian,  $\mathcal{R}$  is given by an orthogonal projection in  $\mathcal{M} \overline{\otimes} \mathcal{M}_{\text{op}}$ . Here, we generalize such result to general von Neumann algebras, proving that quantum relations are in bijective correspondence with weak-\* closed left ideals inside  $\mathcal{M} \otimes_{eh} \mathcal{M}$ , where  $\otimes_{eh}$  is the extended Haagerup tensor product. The correspondence between the two is given by identifying  $\mathcal{M} \otimes_{eh} \mathcal{M}$  with  $\mathcal{M}'$ -bimodular operators and proving a double annihilator relation

Given an action of a group/quantum group on  $\mathcal{M}$  we give a definition for invariant quantum relations and prove that in the case of group von Neumann algebras  $\mathcal{L}G$ , invariant quantum relations are left ideals in the measure algebra  $MG$ . At the end we explore possible applications to noncommutative harmonic analysis, in particular noncommutative Gaussian bounds.

## 1. Prerequisites

**1.1. Weaver's Quantum Relations.** In [Wea12, KW12] Kuperberg and Weaver introduced the concept of a quantum relation over a von Neumann algebra  $\mathcal{M} \subset \mathcal{B}(H)$ . They defined a quantum relation to be a weak-\* closed operator bimodule over  $\mathcal{M}'$ , i.e.: a linear weak-\* closed subset  $\mathcal{V} \subset \mathcal{B}(H)$  satisfying that  $\mathcal{M}'\mathcal{V}\mathcal{M}' \subset \mathcal{V}$ . It is easy to see that such notion doesn't depend on the representation  $\mathcal{M} \subset \mathcal{B}(H)$ .

In the case  $\mathcal{M} = \ell_\infty(X) \subset \mathcal{B}(\ell_2 X)$  acting by multiplication operators we have that  $\mathcal{M}' = \mathcal{M}$ . Identifying  $\mathcal{B}(\ell_2 X)$  with matrices indexed by  $X \times X$ , gives that  $\mathcal{V} \subset \mathcal{B}(\ell_2 X)$  is a quantum relation whenever

$$(1.1) \quad [a_{xy}]_{x,y \in X} \in \mathcal{V} \implies [b_x a_{xy} c_y]_{x,y \in X} \in \mathcal{V},$$

for every  $(b_x)_{x \in X}, (c_x)_{x \in X}$ . This in turns easily implies, see [Wea12, Proposition 1.3], that there is a unique subset  $R \subset X \times X$  such that

$$\mathcal{V}_R = \{[a_{xy}]_{x,y} : (x,y) \notin R \implies a_{xy} = 0\}.$$

and reciprocally every such subset  $R \subset X \times X$  have associated the operator bimodule of all matrices supported on  $R$ . When  $\mathcal{M} = L_\infty(X) \subset \mathcal{B}(L_2 X)$  is abelian but not atomic we do not have a bijective correspondence between  $\mathcal{M}$  bimodules and measurable subsets of  $X \times X$ . In that case the natural object to substitute

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the (discrete) relations  $R \subset X \times X$  will be the, so called, measurable relations, i.e. weak-\* open subsets  $\mathcal{R} \subset \mathcal{P}(\mathcal{M}) \times \mathcal{P}(\mathcal{M})$  satisfying that

$$\left( \bigvee_{\alpha} P_{\alpha}, \bigvee_{\beta} Q_{\beta} \right) \in \mathcal{R} \iff \exists \alpha_0, \beta_0 (P_{\alpha_0}, Q_{\beta_0}) \in \mathcal{R}.$$

The measurable relation associated with a quantum relation  $\mathcal{V} \subset \mathcal{B}(L_2(X))$  is given by

$$(1.2) \quad \mathcal{R}_{\mathcal{V}} = \{(P, Q) \in \mathcal{P}(\mathcal{M}) \times \mathcal{P}(\mathcal{M}) : P\mathcal{V}Q \neq \{0\}\}.$$

Notice that in the abelian discrete case we have that  $\mathcal{R}$  is just the set of projections  $(\chi_A, \chi_B)$  such that there are  $x \in A$  and  $y \in B$  with  $(x, y) \in R$ . Reciprocally, given any measurable relation  $\mathcal{R}$  we can associate a quantum relation over  $\mathcal{M}$  given by

$$(1.3) \quad \mathcal{V}_{\mathcal{R}} = \{T \in \mathcal{B}(L_2 X) : PTQ = 0, \forall (P, Q) \notin \mathcal{R}\}.$$

It is proved in [Wea12] that the map  $\mathcal{R} \mapsto \mathcal{V}_{\mathcal{R}}$  is injective. Unfortunately it is not surjective in general. This has to do with the fact that all the operator bimodules  $\mathcal{V}$  arising like in 1.3 are not just weak-\* closed but *operator reflexive*, see [Erd86, Lar82] and in particular closed in the weak operator topology, or WOT in short. The way to fix that is to observe that if  $\mathcal{V} \subset \mathcal{B}(H)$  is any weak-\* closed linear subspace  $\mathbf{1} \otimes \mathcal{V} \subset \mathcal{B}(\ell_2 \otimes_2 H)$  is operator reflexive. Since  $\mathbf{1} \otimes \mathcal{V}$  is a  $\mathbb{C}\mathbf{1} \otimes \mathcal{M}'$ -bimodule and  $(\mathbb{C}\mathbf{1} \otimes \mathcal{M}')' = \mathcal{B}(\ell_2) \overline{\otimes} \mathcal{M}$  we have that  $\mathbf{1} \otimes \mathcal{V}$  is a quantum relation over the amplified algebra  $\mathcal{B}(\ell_2) \overline{\otimes} \mathcal{M}$ . This suggests that the right definition for quantum relations as pairs of related projections is given by amplified projections in  $\mathcal{B}(\ell_2) \overline{\otimes} \mathcal{M}$ . The next definition captures this intuition.

**Definition 1.1.** ([Wea12, Definition 2.24]) We will say that  $\mathcal{R} \subset \mathcal{P}(\mathcal{M} \overline{\otimes} \mathcal{B}(\ell_2)) \times \mathcal{P}(\mathcal{M} \overline{\otimes} \mathcal{B}(\ell_2))$  is an *intrinsic quantum relation* iff

- (i)  $\mathcal{R}$  is weak-\* open.
- (ii)  $(0, 0) \notin \mathcal{R}$ .
- (iii) If  $(P_{\alpha})_{\alpha \in A}$  and  $(Q_{\beta})_{\beta \in B}$  are sets of families of projections in  $\mathcal{P}(\mathcal{M} \overline{\otimes} \mathcal{B}(\ell_2))$  then

$$\left( \bigvee_{\alpha \in A} P_{\alpha}, \bigvee_{\beta \in B} Q_{\beta} \right) \in \mathcal{R} \iff \exists \alpha_0 \in A, \beta_0 \in B \text{ such that } (P_{\alpha_0}, Q_{\beta_0}) \in \mathcal{R}.$$

- (iv) For every  $B \in \mathbf{1} \otimes \mathcal{B}(\ell_2)$  we have that

$$([BP], Q) \in \mathcal{R} \iff (P, [B^*Q]) \in \mathcal{R},$$

where  $[A]$  represents the left (or final) projection of the operator  $A$ .

Quantum relations over  $\mathcal{M} \subset \mathcal{B}(H)$  and intrinsic quantum relations (or i.q.r.) over  $\mathcal{M}$  are in bijective correspondence and the adaptations of the maps 1.2 and 1.3 are inverse of each other. Indeed, this correspondence works for every von Neumann algebra  $\mathcal{M} \subset \mathcal{B}(H)$  not necessarily abelian or discrete, see [Wea12, Theorem 2.32].

Through this article we are going to employ liberally the language of operator spaces, see [Pis03, ER00, BM04] for more information. An operator space is a closed linear subset  $E \subset \mathcal{B}(H)$ . Given two operator spaces  $E \subset \mathcal{B}(H_1)$  and  $F \subset \mathcal{B}(H_2)$  we say that a linear map  $\phi : E \rightarrow F$  is *completely bounded*, or *c.b.* in short, iff the matrix amplifications  $\text{Id} \otimes \phi : M_n[E] \subset \mathcal{B}(\ell_2^n \otimes_2 H_1) \rightarrow M_n[F] \subset \mathcal{B}(\ell_2^n \otimes_2 H_2)$

are uniformly bounded on  $n$ . We are going to denote by  $\mathcal{CB}(E, F)$  the space of all completely bounded (or c.b.) operators with the norm given by

$$\|\phi\|_{\text{cb}} = \sup_{n \geq 1} \{\|\text{Id} \otimes \phi : M_n[E] \rightarrow M_n[F]\|\}.$$

The category of operator spaces is the collection of all operator spaces with c.b. maps as morphisms. There is also an intrinsic characterization of operator spaces as Banach spaces endowed with collections of matrix norms satisfying the Ruan's axioms. Either an isometric injection  $j : E \rightarrow \mathcal{B}(H)$  or a family of compatible matrix norm will be called an *operator space structure*, or *o.s.s.* in short.

Let  $X$  be a discrete measure space with the counting measure and let us identify  $\mathcal{B}(\ell_2 X)$  with matrices indexed by  $X$ . Given a matrix  $m = [m_{xy}]_{x, y \in X}$  we define the *Schur multiplier* of symbol  $m$  as the operator  $S_m$  given by

$$S_m([a_{xy}]) = [m_{xy} a_{xy}].$$

Whenever  $S_m$  is completely bounded we will say that  $S_m$  is a c.b. Schur multiplier. We are going to denote by  $\mathfrak{M}(X) \subset \mathcal{CB}(\mathcal{B}(\ell_2 X))$  the set of all c.b. Schur multipliers and by  $\mathfrak{M}^\sigma(X)$  the space of all c.b. and normal ones (i.e. weak-\* continuous for  $S_1(\ell_2 X)^* = \mathcal{B}(\ell_2 X)$ ). Assume that  $X$  is a finite set, let  $R \subset X \times X$  be a relation and  $\mathcal{V} \subset \mathcal{B}(\ell_2 X)$  be its associated quantum relation. We have that the ideal  $J \subset \mathfrak{M}^\sigma(X) = \mathfrak{M}(X)$  given by

$$(1.4) \quad J = \{S \in \mathfrak{M}(X) : S|_{\mathcal{V}} = 0\}$$

contains just the Schur multipliers  $S_m$  whose symbol  $m$  satisfies that  $m_{xy} = 0$  if  $(x, y) \in R$ . The reciprocal is also true and we have the following.

**Proposition 1.1.** *Let  $X$  be a finite set and  $\ell_\infty(X) \subset \mathcal{B}(\ell_2 X)$  and  $\mathcal{V} \subset \mathcal{B}(\ell_2 X)$  be as above. Then if  $J$  an ideal in  $\mathfrak{M}^\sigma(X)$  we have that*

$$\begin{aligned} \mathcal{V}_J &= \{T \in \mathcal{B}(\ell_2^n) : S(T) = 0, \forall S \in J\} \\ J_{\mathcal{V}} &= \{S \in \mathfrak{M}^\sigma(X) : S|_{\mathcal{V}} = 0\} \end{aligned}$$

*are bijections between the sets of quantum relations and the set of ideals of Schur multipliers. Furthermore, the maps  $\mathcal{V} \mapsto J_{\mathcal{V}}$  and  $J \mapsto \mathcal{V}_J$  are inverse of each other.*

Such result was generalized to general, not necessarily abelian, finite dimensional von Neumann algebras  $\mathcal{M} \subset \mathcal{B}(H)$  by Weaver [Wea12]. For that end recall that  $\mathfrak{M}^\sigma(X)$  is actually equal to the algebra of all completely bounded normal operators  $S : \mathcal{B}(\ell_2 X) \rightarrow \mathcal{B}(\ell_2 X)$  that are  $\ell_\infty(X)$ -bimodular. We are going to denote the the algebras of  $\mathcal{M}'$ -bimodular c.b. normal operators on  $\mathcal{B}(H)$  by  $\mathcal{CB}_{\mathcal{M}', \mathcal{M}'}^\sigma(\mathcal{B}(H))$ . It is trivial to see that in the case of finite dimensional  $\mathcal{M}$  we have a bounded, quasi-isometric and multiplicative map  $\Phi : \mathcal{M} \otimes_{\min} \mathcal{M}_{\text{op}} \rightarrow \mathcal{CB}_{\mathcal{M}', \mathcal{M}'}^\sigma(\mathcal{B}(H))$  given by extension of

$$(1.5) \quad \Phi(x \otimes y) = (T \mapsto xTy).$$

To see that, let  $n = \dim \mathcal{M}$ , so that,  $\mathcal{B}(H)$  is quasi-isometric to  $\ell_2^n \otimes \ell_2^n$  and  $\mathcal{CB}^\sigma(\mathcal{B}(H))$  is quasi isometric to  $\mathcal{B}(\ell_2 \otimes \ell_2)$ . If  $x, y \in \mathcal{M}'$ , we denote by  $T_{xy}$  the operator given by  $S \mapsto xSy$ . It is clear that  $\phi \in \mathcal{B}(\ell_2 \otimes \ell_2)$  is  $\mathcal{M}'$ -bimodular iff it belongs to the commutant of  $\{T_{xy}\}_{x, y \in \mathcal{M}'}$  but such algebra is isomorphic to  $\mathcal{M} \otimes \mathcal{M}_{\text{op}}$  as we claimed. If  $\mathcal{V} \subset \mathcal{B}(H)$  is a quantum relation over  $\mathcal{M}$  we have that  $J_{\mathcal{V}} = \{s \in \mathcal{M} \otimes \mathcal{M}_{\text{op}} : \Phi_s|_{\mathcal{V}} = 0\}$  is a left ideal and therefore is of the form

$J_{\mathcal{V}} = (\mathcal{M} \otimes \mathcal{M}_{\text{op}})p_{\mathcal{V}}$  for some  $p_{\mathcal{V}} \in \mathcal{P}(\mathcal{M} \otimes \mathcal{M}_{\text{op}})$ . Furthermore, we have the following.

**Proposition 1.2.** ([Wea12, Proposition 2.23]) *If  $\mathcal{M}$  is finite dimensional the correspondence  $\mathcal{V} \mapsto p_{\mathcal{V}}^{\perp}$  defined as above is an order-preserving bijection between quantum relations over  $\mathcal{M}$  and projections in  $\mathcal{M} \otimes \mathcal{M}_{\text{op}}$ .*

In the case of infinite dimensional von Neumann algebras  $\mathcal{M}$  the result above fails and not every quantum relation can be associated with a projection in  $\mathcal{M} \overline{\otimes} \mathcal{M}_{\text{op}}$ . The reason for that is that although the map  $\Phi : \mathcal{M} \otimes \mathcal{M}_{\text{op}} \rightarrow \mathcal{CB}_{\mathcal{M}' \mathcal{M}'}(\mathcal{B}(H))$  is bounded and multiplicative for every finite dimensional algebra  $\mathcal{M}$  it is far from isometric. Indeed its norm explodes with  $n = \dim(\mathcal{M})$ . The problem can be solved by changing the tensor norm from the spatial tensor norm to the Haagerup tensor norm of the two von Neumann algebras. With that tool at hand we will be able to prove a generalization of 1.1 for general algebras in the next section.

**1.2. Module Maps and The Haagerup Tensor Product.** Let  $E, F$  be two operator spaces. We define the bilinear form  $\odot : M_n[E] \times M_n[F] \rightarrow M_n[E \otimes_{\text{alg}} F]$  by

$$[x_{ij}] \odot [y_{ij}] = \left[ \sum_{k=1}^n x_{ik} \otimes y_{kj} \right]_{i,j}.$$

Of course such definition makes perfect sense with matrices of different sizes  $\odot : M_{n,m}[E] \times M_{m,l}[F] \rightarrow M_{n,l}[E \otimes_{\text{alg}} F]$  just by embedding all matrices inside  $M_{\max\{n,m,l\}}$  and restricting. The *Haagerup tensor norm* for  $z \in E \otimes_{\text{alg}} F$  is defined to be

$$\begin{aligned} \|z\|_h &= \inf \{ \|u\|_{M_{1,n}(E)} \|v\|_{M_{1,n}(F)} : z = u \odot v \} \\ &= \inf \left\{ \left\| \sum_{k=1}^n x_k x_k^* \right\|^{\frac{1}{2}} \left\| \sum_{k=1}^n y_k^* y_k \right\|^{\frac{1}{2}} : z = \sum_{k=1}^n x_k \otimes y_k \right\} \end{aligned}$$

The Haagerup tensor product  $E \otimes_h F$  is defined as the completion under that norm. Similarly  $E \otimes_h F$  can be given an o.s.s by defining:

$$\|x\|_{M_n(E \otimes_h F)} = \inf \{ \|u\|_{M_{n,k}(E)} \|v\|_{M_{k,n}(F)} : z = u \odot v \}.$$

In the case of two dual operator spaces  $E^*$  and  $F^*$  the *weak-\* Haagerup tensor product*, introduced in [BS92] by Blecher and Smith, is given by

$$E^* \otimes_{w^*h} F^* = (E \otimes_h F)^*.$$

Since the Haagerup tensor norm is self dual, see [ER91], we have that  $E^* \otimes_h F^*$  embeds inside  $E^* \otimes_{w^*h} F^*$  isometrically and is weak-\* dense. This tensor product is a complemented subspace of the *normal Haagerup tensor product*  $E \otimes_{\sigma h} F$  introduced by Effros and Kishimoto [EK87] and which satisfies that

$$(E \otimes_h F)^{**} = (E^{**} \otimes_{\sigma h} F^{**}).$$

In [ER03] Effros and Ruan introduced the *extended Haagerup tensor product* generalizing the weak-\* Haagerup tensor to (potentially) non-dual operator spaces. Indeed if  $x = [x_{ij}]_{i,j}$  is a matrix whose entries are, possibly infinite, sums of simple tensors, we say that  $x \in M_m(E \otimes_{eh} F)$  iff

$$\|x\|_{M_m(E \otimes_{eh} F)} = \inf \{ \|u\|_{M_{m,I}(E)} \|v\|_{M_{m,I}(F)} : x = u \odot v \}$$

for every possible index set  $I$ . It can be seen that it is enough to take  $I$  to be the smallest cardinality of a dense set in  $H$  with  $E, F \subset \mathcal{B}(H)$ . Particularly when  $E$

and  $F$  are separable von Neumann algebras we can take  $I$  numerable. In the case of  $E^*$ ,  $F^*$  being dual operator spaces, we have that

$$\begin{aligned} E^* \otimes_{w^*h} F^* &= E^* \otimes_{eh} F^* \\ E^* \otimes_{\sigma h} F^* &= (E \otimes_{eh} F)^*. \end{aligned}$$

The coarser topology in  $E^* \otimes_{eh} F^*$  making the pairing with every element in  $\mathcal{M}_* \otimes_{eh} \mathcal{M}_*$  continuous is strictly finer than the weak-\* topology given by the predual  $\mathcal{M}_* \otimes_h \mathcal{M}_*$ . Since  $E^* \otimes_{eh} F^* \subset E^* \otimes_{\sigma h} F^*$  is  $\sigma(E^* \otimes_{eh} F^*)$ -closed,  $E^* \otimes_{eh} F^*$ , with the  $\sigma(E^* \otimes_{eh} F^*)$  topology, is a dual space. Its predual is obtained by a quotient of  $E^* \otimes_{eh} F^*$

When  $E = \mathcal{N}$ ,  $F = \mathcal{M}$  are von Neumann algebras  $\mathcal{N} \otimes_{eh} \mathcal{M}$  is a weak-\* Banach algebra with a jointly completely bounded multiplication, see [ER00, pp. 126], given by extension of

$$(x \otimes y)(z \otimes t) = xz \otimes ty.$$

When  $\mathcal{M} = \mathcal{N}$  there is also a natural multiplicative involution  $(x \otimes y)^\dagger = y^* \otimes x^*$ .

Recall that the space of completely bounded  $\mathcal{CB}(E, F)$  has a natural o.s.s. given by the identification  $M_n(\mathcal{CB}(E, F)) = \mathcal{CB}(E, M_n(F))$ . If  $E^*$  and  $F^*$  are dual operator spaces we define  $\mathcal{CB}^\sigma(E^*, F^*) \subset \mathcal{CB}(E^*, F^*)$  to be subspace of all weak-\* continuous operators. We have a natural identification  $\mathcal{CB}^\sigma(E^*, F^*) = \mathcal{CB}(F, E)$ . When  $E, F \subset \mathcal{B}(H)$  are bimodules over a von Neumann algebra  $\mathcal{M} \subset \mathcal{B}(H)$  we will denote by  $\mathcal{CB}_{\mathcal{M}\mathcal{M}}(E, F)$  and  $\mathcal{CB}_{\mathcal{M}\mathcal{M}}^\sigma(E, F)$  the subspaces of completely bounded and bimodular operators. Such subspaces are easily seen to be norm closed. We will treat mainly the case when  $E = F = \mathcal{B}(H)$ . We have, using that  $\mathcal{K}(H)^{**} = \mathcal{B}(H)$  and that  $\mathcal{CB}(E, F^*) = \mathcal{CB}^\sigma(E^{**}, F^*)$ , see [BM04, (1.28)], that

$$(1.6) \quad \mathcal{CB}^\sigma(\mathcal{B}(H)) = \mathcal{CB}(\mathcal{K}(H), \mathcal{B}(H)).$$

The identification is given by restriction to  $\mathcal{K}(H) \subset \mathcal{B}(H)$  and by passage to the second dual. The identity 1.6 allow us to give a predual for  $\mathcal{CB}^\sigma(\mathcal{B}(H))$  by

$$(1.7) \quad \begin{aligned} \mathcal{CB}(\mathcal{K}(H), \mathcal{B}(H)) &= \mathcal{CB}(\mathcal{K}(H), \mathbb{C}) \otimes_{\mathcal{F}} \mathcal{B}(H) \quad (\text{by [Pis03, Th. 4.1]}) \\ &= (\mathcal{K}(H) \widehat{\otimes} S_1(H))^*, \end{aligned}$$

where  $\otimes_{\mathcal{F}}$  is the Fubini tensor product, see [EKR93], [ER03] or [ER00] which is isomorphic to the dual of the (operator space) projective tensor product  $\widehat{\otimes}$ , see [ER00, Chap. 7]. Similarly the predual of  $\mathcal{CB}(\mathcal{B}(H))$  is given by  $\mathcal{B}(H) \widehat{\otimes} S_1(H)$ . In both cases the pairing is given by linear extension of  $\langle T \otimes \xi, \Psi \rangle = \langle \xi, \Psi(T) \rangle$ , for  $\Psi \in \mathcal{CB}(\mathcal{B}(H))$ . A subtle point is that the coarser topology in  $\mathcal{CB}^\sigma(\mathcal{B}(H))$  making the pairing with all the elements in  $\mathcal{B}(H) \widehat{\otimes} S_1(H)$  continuous is, in general, strictly finer than the weak-\* topology given by the predual  $\mathcal{K}(H) \widehat{\otimes} S_1(H)$ . To see that, notice that the following inclusion holds

$$\mathcal{K}(H) \widehat{\otimes} S_1(H) \subset \mathcal{B}(H) \widehat{\otimes} S_1(H).$$

Indeed, the inclusion above is just a consequence of the fact that  $\mathcal{K}(H) \subset \mathcal{B}(H)$  and the injectivity of the functor  $E \mapsto \mathcal{M}_* \widehat{\otimes} E$ , where  $\mathcal{M}_*$  is the predual of any *hyperfinit* von Neumann algebra, see [Pis98]. Since  $\sigma(\mathcal{K}(H) \widehat{\otimes} S_1(H))$ -closed sets are  $\sigma(\mathcal{B}(H) \widehat{\otimes} S_1(H))$ -closed we have that  $\mathcal{CB}^\sigma(\mathcal{B}(H)) \subset \mathcal{CB}(\mathcal{B}(H))$  is  $\sigma(\mathcal{B}(H) \widehat{\otimes} S_1(H))$ -closed and so the  $\sigma(\mathcal{B}(H) \widehat{\otimes} S_1(H))$  topology induces another predual for  $\mathcal{CB}^\sigma(\mathcal{B}(H))$ . Clearly, the topology of pointwise weak-\* convergence in  $\mathcal{CB}^\sigma(\mathcal{B}(H))$  is coarser than the  $\sigma(\mathcal{B}(H) \widehat{\otimes} S_1(H))$  topology. Analogously, the topology of pointwise (in  $\mathcal{K}(H)$ )

weak-\* topology is coarser than the  $\sigma(\mathcal{K}(H) \widehat{\otimes} S_1(H))$  topology. In both cases the topologies coincide over bounded sets.

The subspace of bimodular operators  $\mathcal{CB}_{\mathcal{M}\mathcal{M}}^\sigma(\mathcal{B}(H))$  is closed in both the  $\sigma(\mathcal{K}(H) \widehat{\otimes} S_1(H))$  and the  $\sigma(\mathcal{B}(H) \widehat{\otimes} S_1(H))$  topologies. Indeed, it is closed in the  $\mathcal{K}(H)$ -pointwise weak-\* topology which is coarser than both. As a consequence, using the Hahn-Banach Theorem, we get that  $\mathcal{CB}_{\mathcal{M}\mathcal{M}}^\sigma(\mathcal{B}(H))$  inherits two natural predual topologies

$$\begin{aligned}\mathcal{CB}_{\mathcal{M}\mathcal{M}}^\sigma(\mathcal{B}(H)) &= (\mathcal{B}(H) \widehat{\otimes} S_1(H)/K_2)^*, \\ \mathcal{CB}_{\mathcal{M}\mathcal{M}}^\sigma(\mathcal{B}(H)) &= (\mathcal{K}(H) \widehat{\otimes} S_1(H)/K_1)^*,\end{aligned}$$

where  $K_1, K_2$  are the corresponding preannihilators. Similarly  $\mathcal{CB}_{\mathcal{M}\mathcal{M}}(\mathcal{B}(H))$  is also a dual space with the  $\sigma(\mathcal{B}(H) \widehat{\otimes} S_1(H))$  topology. The spaces  $\mathcal{CB}_{\mathcal{M}\mathcal{M}}(\mathcal{K}(H))$ ,  $\mathcal{CB}_{\mathcal{M}\mathcal{M}}^\sigma(\mathcal{B}(H))$  and  $\mathcal{CB}_{\mathcal{M}\mathcal{M}}(\mathcal{B}(H))$  are Banach algebras with the composition operation. They have a natural multiplicative involution given by  $\Psi_1^\dagger(T) = \Psi_1(T^*)^*$  and satisfying that  $(\Psi_1\Psi_2)^\dagger = \Psi_1^\dagger\Psi_2^\dagger$ .

**Example 1.2.** Recall that in the case of  $\mathcal{M} = \ell_\infty(X) \subset \mathcal{B}(\ell_2 X)$  we have that

$$\begin{aligned}\mathcal{CB}_{\ell_\infty(X)\ell_\infty(X)}^\sigma(\mathcal{B}(\ell_2 X)) &= \mathfrak{M}^\sigma(X), \\ \mathcal{CB}_{\ell_\infty(X)\ell_\infty(X)}(\mathcal{B}(\ell_2 X)) &= \mathfrak{M}(X).\end{aligned}$$

For non-discrete measure spaces  $(X, \mu)$  we have that  $\mathcal{CB}_{L_\infty(X)L_\infty(X)}^\sigma(\mathcal{B}(L_2 X))$  corresponds to the algebra of *measurable Schur multipliers*, see [Spr04].

Now we are in position of stating the isomorphism between Haagerup tensors and bimodular operators.

**Theorem 1.3.** *Let  $\mathcal{M} \subset \mathcal{B}(H)$  be a von Neumann algebra. The map  $\Phi$  defined by  $x \otimes y \mapsto \Phi_{x \otimes y}$ , where*

$$\Phi_{x \otimes y}(T) = x T y,$$

*extends to a surjective complete isometry and a  $\dagger$ -preserving homomorphism between the following spaces*

- (i)  $\Phi : \mathcal{M} \otimes_h \mathcal{M} \rightarrow \mathcal{CB}_{\mathcal{M}'\mathcal{M}'}(\mathcal{K}(H)).$
- (ii)  $\Phi : \mathcal{M} \otimes_{eh} \mathcal{M} \rightarrow \mathcal{CB}_{\mathcal{M}'\mathcal{M}'}^\sigma(\mathcal{B}(H)).$
- (iii)  $\Phi : \mathcal{M} \otimes_{\sigma h} \mathcal{M} \rightarrow \mathcal{CB}_{\mathcal{M}'\mathcal{M}'}(\mathcal{B}(H)).$

*Furthermore, the map in (iii) is  $\sigma(\mathcal{M}_* \otimes_{eh} \mathcal{M}_*)$  to  $\sigma(\mathcal{B}(H) \widehat{\otimes} S_1(H))$  continuous and the map in (ii) is both  $\sigma(\mathcal{M}_* \otimes_h \mathcal{M}_*)$  to  $\sigma(\mathcal{K}(H) \widehat{\otimes} S_1(H))$  continuous and  $\sigma(\mathcal{M}_* \otimes_{eh} \mathcal{M}_*)$  to  $\sigma(\mathcal{B}(H) \widehat{\otimes} S_1(H))$  continuous.*

The result above is well known to the experts, although their pieces are scattered throughout the literature. We will just give a brief sketch with references. Recall too that the first appearance of such result is credited to be in an unpublished note of Haagerup [Haa86].

**Proof.** Let us concentrate on (ii), which will be the most important for our applications. The fact that  $\Phi$  is a complete contraction amounts to a trivial calculation.

Indeed, if  $s = \sum_j x_j \otimes y_j$  we may define, for every  $1 \leq n$ , the matrices

$$x = \sum_{i=0}^n \sum_j e_{ij} \otimes x_j, \quad y = \sum_{j=0}^n \sum_i e_{ij} \otimes x_i$$

inside  $\mathcal{B}(\ell_2) \overline{\otimes} \mathcal{M}$ , where  $\{e_{ij}\}$  is a system of matrix units. Then  $(\text{Id}_{M_n} \otimes \Phi_s)(T)$  satisfies that

$$(\text{Id}_{M_n} \otimes \Phi_s)(T) = P_n x (1 \otimes T) y P_n,$$

where  $P_n$  is the orthogonal projection on the span of  $\{e_j\}_{j \leq n}$ . Clearly

$$\|\text{Id}_{M_n} \otimes \Phi_s\| \leq \|x^* x\|_{\mathcal{M}}^{\frac{1}{2}} \|y y^*\|_{\mathcal{M}}^{\frac{1}{2}}$$

and  $\Phi$  is an  $\mathcal{M}'$ -bimodular operator. Taking the supremum over  $n \geq 1$  and the infimum over all representations of  $s$  gives that  $\|\Phi_s\|_{\text{cb}} \leq \|s\|_{\mathcal{M} \otimes_{eh} \mathcal{M}}$ . To see that it is surjective notice that if  $\Psi \in \mathcal{CB}_{\mathcal{M}', \mathcal{M}'}^{\sigma}(\mathcal{B}(H)) = \mathcal{CB}_{\mathcal{M}', \mathcal{M}'}(\mathcal{K}(H), \mathcal{B}(H))$  by Wittstock's factorization theorem for c.b. maps, see [Pau86], we have that there is a large enough  $\ell_2$  (we can take the dimension of  $\ell_2$  to be equal to that of  $H$  for infinite dimensional spaces), a representation  $\pi : \mathcal{K}(H) \rightarrow \mathcal{B}(\ell_2 \otimes_2 H)$  and two elements  $x \in \mathcal{B}(\ell_2 \otimes_2 H, H)$ ,  $y \in \mathcal{B}(H, \ell_2 \otimes_2 H)$  such that  $\Psi(x) = x \pi(x) y$  and  $\|\Psi\|_{\text{cb}} = \|x\| \|y\|$  but we can identify  $x$  and  $y$  with a row and a column respectively inside  $\mathcal{B}(\ell_2) \overline{\otimes} \mathcal{B}(H)$  and we have that  $\Psi = \Phi_s$ , where  $s = x \odot y \in \mathcal{B}(H) \otimes_{eh} \mathcal{B}(H)$ . It only rest to prove that if  $\Psi$  is  $\mathcal{M}'$ -bimodular we can pick  $x, y \in \mathcal{B}(\ell_2) \overline{\otimes} \mathcal{M}$ , which is the main result in [Smi91, Theorem 3.1]. The rest of the points are similarly proved, see also [BS92] for (iii).  $\square$

As a consequence of the preceding theorem we are going to identify at times  $\mathcal{M} \otimes_{eh} \mathcal{M}$  and its weak-\* topology with  $\mathcal{CB}_{\mathcal{M}', \mathcal{M}'}^{\sigma}(\mathcal{B}(H))$  and  $\sigma(\mathcal{B}(H) \widehat{\otimes} S_1(H))$ . The following lemma describe the weak-\* continuous functionals on  $\mathcal{M} \otimes_{eh} \mathcal{M}$  for its different preduals.

**Lemma 1.4.** *Let  $\phi \in (\mathcal{M} \otimes_{eh} \mathcal{M})^*$ , then*

(i)  *$\phi$  is  $\sigma(\mathcal{M}_* \otimes_{eh} \mathcal{M}_*)$ -continuous iff*

$$\langle \phi, s \rangle = \langle C, (\text{Id} \otimes \Phi_s)(B) \rangle,$$

*where  $C \in S_1(\ell_2 \otimes_2 H)$  and  $B \in \mathcal{B}(\ell_2 \otimes_2 H)$*

(ii)  *$\phi$  is  $\sigma(\mathcal{M}_* \otimes_h \mathcal{M}_*)$ -continuous iff*

$$\langle \phi, s \rangle = \langle C, (\text{Id} \otimes \Phi_s)(B) \rangle,$$

*where  $C \in S_1(\ell_2 \otimes_2 H)$  and  $B \in \mathcal{K}(\ell_2 \otimes_2 H)$*

Furthermore,  $\phi$  is pointwise weak-\* continuous, iff  $B$  in (i) can be taken in  $\mathcal{B}(\ell_2^n \otimes_2 H)$ . Similarly,  $\phi$  is  $\mathcal{K}(H)$ -pointwise weak-\* continuous iff we can take  $B \in \mathcal{K}(\ell_2^n \otimes_2 H)$ .

**Proof.** We will prove (i) first. Since, by Theorem 1.3, the predual for the  $\sigma(\mathcal{M}_* \otimes_h \mathcal{M}_*)$  topology is given by  $(\mathcal{M} \otimes_{eh} \mathcal{M})_* = (\mathcal{K}(H) \widehat{\otimes} S_1(H))/F$ , where  $F$  is the preannihilator of the  $\mathcal{M}'$ -bimodular maps,  $\phi$  can be lifted to an element (that we will denote also by  $\phi$ ) in  $\mathcal{K}(H) \widehat{\otimes} S_1(H)$  inducing the same functional. By definition of the o.s. projective tensor product we have that there are, possibly

infinite, index sets  $I_1, I_2$  and elements  $A \in \mathcal{K}_{I_1} \otimes_{\min} S_1(H)$ ,  $B \in \mathcal{K}_{I_2} \otimes_{\min} \mathcal{K}(H)$  and  $\alpha, \beta \in S_2(\ell_2^{I_1}, \ell_2^{I_2})$ , where  $\mathcal{K}_{I_i} = \mathcal{K}(\ell_2^{I_i})$ , such that

$$\phi = \sum_{i,j \in I_1, p,q \in I_2} \alpha_{ip} (B_{ij} \otimes A_{pq}) \beta_{jq}.$$

The action on  $s \in \mathcal{M} \otimes_{eh} \mathcal{M}$  is given by

$$\begin{aligned} \langle \phi, s \rangle &= \sum_{i,j \in I_1, p,q \in I_2} \alpha_{ip} \langle A_{ij}, \Phi_s(B_{pq}) \rangle \beta_{jq} \\ &= \sum_{p,q \in I_2} \left\langle \sum_{i,j \in I_1} \bar{\alpha}_{ip} A_{ij} \beta_{jq}, \Phi_s(B_{pq}) \right\rangle \\ &= \langle (\alpha^* \otimes \mathbf{1}) A (\beta \otimes \mathbf{1}), (\text{Id}_{\mathcal{K}_{I_2}} \otimes \Phi_s)(B) \rangle. \end{aligned}$$

Note that, by [Pis98, Theorem 1.5],  $C = (\alpha^* \otimes \mathbf{1}) A (\beta \otimes \mathbf{1}) \in S_1(\ell_2^{I_2})[S_1(H)] \simeq S_1(\ell_2^{I_2} \otimes_2 H)$ . We have thus that every weak-\* continuous functional  $\phi$  can be expressed as

$$\langle \phi, s \rangle = \langle C, (\text{Id}_{\mathcal{K}} \otimes \Phi_s)(B) \rangle,$$

concluding the proof of (i). The same techniques yield (ii).

The other claims in the statement follows by a repetition of the ideas used to prove that SOT-continuous and WOT-continuous functionals coincide over  $\mathcal{B}(H)$ . Indeed, assume  $\phi$  is pointwise weak-\* continuous. Then, there are finite collection  $T_1, \dots, T_m \in \mathcal{B}(H)$  and  $\xi_1, \xi_2, \dots, \xi_m \in S_1(H)$  such that  $|\phi(\Psi)| < 1$  whenever  $|\langle \xi_i, \Psi(T_i) \rangle| < \epsilon$  for  $i \in \{1, 2, \dots, m\}$ . In particular, taking  $\Psi' = \Psi / \max\{|\langle \xi_i, \Psi(T_i) \rangle|\}$  gives

$$|\phi(\Psi)| \leq \epsilon^{-1} \max\{|\langle \xi_i, \Psi(T_i) \rangle|\} \leq \epsilon^{-1} \sum_{i=1}^m |\langle \xi_i, \Psi(T_i) \rangle|.$$

As a consequence, if  $\Psi(T_i) = 0$ , for  $i \in \{1, 2, \dots, m\}$ , we have  $\phi(\Psi) = 0$  and so  $\phi$  factors through a finite dimensional space. Therefore,  $\phi$  can be expressed as a finite combination of simple tensors.  $\square$

## 2. The Correspondence Between Ideals and Modules

In this section we are going to prove the correspondence between left ideals in  $\mathcal{M} \otimes_{eh} \mathcal{M}$  and quantum relations over  $\mathcal{M}$ . We are going to start recalling two easy lemmas that will be thoroughly used in this section. The first asserts that the bilinear form  $\odot$  can be extended from  $M_n[\mathcal{M}]$  to  $\mathcal{B}(\ell_2) \overline{\otimes} \mathcal{M}$ , where  $\overline{\otimes}$  is the weak-\* closed spatial tensor or equivalently, since  $\mathcal{B}(\ell_2)$  is a von Neumann algebra, the Fubini tensor product. The second is a stability result for weak-\* closed left ideals in  $\mathcal{M} \otimes_{eh} \mathcal{M}$ . In the forthcoming text we are going to denote by  $\mathcal{B}(\ell_2) \overline{\otimes} (\mathcal{M} \otimes_{eh} \mathcal{M})$  the weak-\* closed tensor product, with respect to the  $\sigma(\mathcal{B}(H) \widehat{\otimes} S_1(H))$  topology. Recall that, using the following identifications

$$\begin{aligned} \mathcal{B}(\ell_2) \overline{\otimes} (\mathcal{M} \otimes_{eh} \mathcal{M}) &\cong \mathcal{B}(\ell_2) \overline{\otimes} \mathcal{CB}_{\mathcal{M}', \mathcal{M}'}^\sigma(\mathcal{B}(H)) \\ &\cong \mathcal{CB}_{\mathcal{M}', \mathcal{M}'}^\sigma(\mathcal{B}(H), \mathcal{B}(\ell_2 \otimes_2 H)) \end{aligned}$$

and reasoning like in (1.7), we have that the predual of  $\mathcal{B}(\ell_2) \overline{\otimes} (\mathcal{M} \otimes_{eh} \mathcal{M})$  can be expressed as a quotient of  $\mathcal{B}(H) \widehat{\otimes} S_1(\ell_2 \otimes_2 H)$ .



**Lemma 2.1.** *The bilinear map  $\odot : \mathcal{B}(\ell_2) \overline{\otimes} \mathcal{M} \times \mathcal{B}(\ell_2) \overline{\otimes} \mathcal{M} \rightarrow \mathcal{B}(\ell_2) \overline{\otimes} (\mathcal{M} \otimes_{eh} \mathcal{M})$  is bounded and continuous over bounded sets if  $\mathcal{B}(\ell_2) \overline{\otimes} \mathcal{M} \times \mathcal{B}(\ell_2) \overline{\otimes} \mathcal{M}$  is given the product strong operator topology (SOT) and  $\mathcal{B}(\ell_2) \overline{\otimes} (\mathcal{M} \otimes_{eh} \mathcal{M})$  the  $\sigma(\mathcal{B}(H) \widehat{\otimes} S_1(H))$  topology.*

**Proof.** Let  $(y_\alpha)_\alpha, (x_\alpha)_\alpha \subset \text{Ball}(\mathcal{B}(\ell_2) \overline{\otimes} \mathcal{M})$  be nets in the unit ball satisfying that  $x_\alpha \rightarrow x$  and  $y_\alpha \rightarrow y$  in the SOT. Since the SOT and  $\sigma$ -SOT topologies agree on bounded set we can assume that we have SOT convergence for any given representation of  $\mathcal{B}(\ell_2) \overline{\otimes} \mathcal{M}$  and in particular for its representation on the Hilbert-Schmidt operators  $S_2(\ell_2 \otimes_2 H)$ . Again since the weak-\* topology and the pointwise weak-\* topology of  $\mathcal{CB}_{\mathcal{M}, \mathcal{M}'}^\sigma(\mathcal{B}(H), \mathcal{B}(\ell_2 \otimes_2 H))$  agree on bounded sets it is enough to see that for any  $S \in \mathcal{B}(H)$  and  $\xi \in S_1(\ell_2 \otimes_2 H)$ ,  $\langle (S \otimes \xi), x_\alpha \odot y_\alpha \rangle \rightarrow \langle (S \otimes \xi), x \odot y \rangle$ . But using that  $\langle (S \otimes \xi), x \odot y \rangle = \langle \xi, x (\mathbf{1} \otimes S) y \rangle$  and expressing  $\xi = \eta \zeta^*$ , where  $\eta, \zeta$  are Hilbert-Schmidt operators, gives  $\langle (S \otimes \xi), x \odot y \rangle = \langle \eta, x (\mathbf{1} \otimes S) y \zeta \rangle$ , where the last paring is just the inner product of  $S_2(\ell_2 \otimes_2 H)$ . Using the SOT-convergence of  $x_\alpha$  and  $y_\alpha$  gives

$$\begin{aligned} & |\langle \eta, x_\alpha (\mathbf{1} \otimes S) y_\alpha - x (\mathbf{1} \otimes S) y \zeta \rangle| \\ & \leq |\langle \eta, x_\alpha (\mathbf{1} \otimes S) (y_\alpha - y) \zeta \rangle| + |\langle \eta, (x_\alpha - x) (\mathbf{1} \otimes S) y \zeta \rangle| \\ & \leq \left( \sup_\alpha \|(\mathbf{1} \otimes S^*) x_\alpha^* \eta\| \right) \|(y_\alpha - y) \zeta\| + \|\eta\| \|(x_\alpha - x) (\mathbf{1} \otimes S) y \zeta\| \\ & \rightarrow 0, \end{aligned}$$

and that concludes the proof.  $\square$

**Lemma 2.2.** *Let  $J \subset \mathcal{M} \otimes_{eh} \mathcal{M}$  be a  $\sigma(\mathcal{B}(H) \widehat{\otimes} S_1(H))$ -closed left ideal, the following holds*

- (i) *If  $X, Y \in \mathcal{B}(\ell_2) \overline{\otimes} \mathcal{M}$  satisfy that  $X \odot Y \in \mathcal{B}(\ell_2) \overline{\otimes} J$  then  $ZX \odot YT \in \mathcal{B}(\ell_2) \overline{\otimes} J$  for every  $Z, T \in \mathcal{B}(\ell_2) \overline{\otimes} \mathcal{M}$ .*
- (ii)  *$X \odot Y \in \mathcal{B}(\ell_2) \overline{\otimes} J$  if and only if  $[X^*] \odot [Y] \in \mathcal{B}(\ell_2) \overline{\otimes} J$ .*

**Proof.** Since  $J \subset \mathcal{M} \otimes_{eh} \mathcal{M}$  is a  $\sigma(\mathcal{B}(H) \widehat{\otimes} S_1(H))$ -closed left ideal,  $J' = \mathcal{B}(\ell_2) \overline{\otimes} J$  is also a  $\sigma(\mathcal{B}(H) \widehat{\otimes} S_1(\ell_2 \otimes_2 H))$ -closed left ideal. Furthermore, it satisfies that  $J'(\mathcal{B}(\ell_2) \otimes \mathbf{1}) = J'$ . For (i) just notice that if  $Z = A \otimes x$ ,  $T = B \otimes y$  are simple tensors, then

$$ZX \odot YT = (A \otimes x \otimes y) (X \odot Y) (B \otimes \mathbf{1}) \in J'$$

Now, approximating  $T$  and  $Z$  by bounded, SOT-convergent nets of sums of simple tensor and applying 2.1 we obtain (i).

For (ii) notice that if  $[X^*] \odot [Y] \in J'$  then, by (i),  $X [X^*] \odot [Y] Y = X \odot Y \in J'$ . For the other implication we just use functional calculus. Indeed, if  $X \odot Y \in J'$  then  $X^* X \odot Y Y^* \in J'$ . Let us denote by  $P = X^* X$  and  $Q = Y Y^*$  and let  $p_n(r)$  be a family of polynomials converging pointwise and boundedly to  $\chi_{[0, \infty)}(r)$ . Then, since all of the powers  $P^n \odot Q^n$  lie in  $J'$  we have that  $p_n(P) \odot p_n(Q) \in J'$ . Since  $p_n(P) \rightarrow \chi_{[0, \infty)}(P) = [X^*]$  and  $p_n(Q) \rightarrow \chi_{[0, \infty)}(Q) = [Y]$  in the SOT, we obtain the claim.  $\square$

We can now prove the main theorem of the section.

**Theorem 2.3.** *Let  $\mathcal{M} \subset \mathcal{B}(H)$  be a von Neumann algebra. The maps*

$$\begin{array}{ccc}
 & \left\{ \begin{array}{l} \mathcal{R} \subset \mathcal{P}(\mathcal{B}(\ell_2) \overline{\otimes} \mathcal{M})^2 : \\ \mathcal{R} \text{ is an i.q.r.} \end{array} \right\} & \\
 \mathcal{V}_{\mathcal{R}} \swarrow & \begin{array}{c} \mathcal{J}_{\mathcal{R}} \quad \mathcal{R}_J \\ \downarrow \quad \uparrow \end{array} & \searrow \mathcal{R}_{\mathcal{V}} \\
 & \left\{ \begin{array}{l} J \subset \mathcal{M} \otimes_{eh} \mathcal{M} : \\ (\mathcal{M} \otimes_{eh} \mathcal{M}) J \subset J, \\ \sigma(\mathcal{B} \widehat{\otimes} S_1)\text{-closed} \end{array} \right\} & \\
 \mathcal{V}_J \swarrow & \begin{array}{c} \mathcal{V}_J \quad J_{\mathcal{V}} \\ \downarrow \quad \uparrow \end{array} & \searrow \\
 & \left\{ \begin{array}{l} \mathcal{V} \subset \mathcal{B}(H) : \\ \mathcal{M}' \mathcal{V} \mathcal{M}' \subset \mathcal{V}, \text{ weak-}* \text{closed} \end{array} \right\} &
 \end{array}$$

given by

$$\begin{aligned}
 \mathcal{J}_{\mathcal{V}} &= \{s \in \mathcal{M} \otimes_{eh} \mathcal{M} : \Phi_s(T) = 0, \forall T \in \mathcal{V}\} \\
 \mathcal{R}_{\mathcal{V}} &= \{(P, Q) \in \mathcal{P}(\mathcal{B}(\ell_2) \overline{\otimes} \mathcal{M})^2 : \exists T \in \mathcal{V}, P(T \otimes \mathbf{1})Q \neq 0\} \\
 \mathcal{V}_J &= \{T \in \mathcal{B}(H) : \Phi_s(T) = 0, \forall s \in J\} \\
 \mathcal{R}_J &= \{(P, Q) \in \mathcal{P}(\mathcal{B}(\ell_2) \overline{\otimes} \mathcal{M})^2 : P \odot Q \notin \mathcal{B}(\ell_2) \overline{\otimes} J\} \\
 J_{\mathcal{R}} &= \overline{\{(\phi \otimes \text{Id})(X \odot Y) : ([X^*], [Y]) \in \mathcal{R}, \phi \in \mathcal{B}(\ell_2)_*^{\text{w}*}\}} \\
 \mathcal{V}_{\mathcal{R}} &= \{T \in \mathcal{B}(\ell_2) : P(T \otimes \mathbf{1})Q = 0, \forall (P, Q) \notin \mathcal{R}\},
 \end{aligned}$$

are well defined, bijective and inverse of each other, i.e.

$$\begin{array}{lll}
 \text{(i)} \quad \mathcal{V}_{J_{\mathcal{V}}} = \mathcal{V} & \text{(iii)} \quad \mathcal{R}_{J_{\mathcal{R}}} = \mathcal{R} & \text{(v)} \quad \mathcal{R}_{\mathcal{V}_{\mathcal{R}}} = \mathcal{R} \\
 \text{(ii)} \quad J_{\mathcal{V}_J} = J & \text{(iv)} \quad J_{\mathcal{R}_J} = J & \text{(vi)} \quad \mathcal{V}_{\mathcal{R}_{\mathcal{V}}} = \mathcal{V}
 \end{array}$$

Furthermore, the rest of the maps commute, giving

$$\begin{array}{lll}
 \text{(1)} \quad \mathcal{R}_{J_{\mathcal{V}}} = \mathcal{R}_{\mathcal{V}} & \text{(3)} \quad \mathcal{V}_{\mathcal{R}_J} = \mathcal{V}_J & \text{(5)} \quad \mathcal{R}_{\mathcal{V}_J} = \mathcal{R}_J \\
 \text{(2)} \quad J_{\mathcal{V}_{\mathcal{R}}} = J_{\mathcal{R}} & \text{(4)} \quad J_{\mathcal{R}_{\mathcal{V}}} = J_{\mathcal{V}} & \text{(6)} \quad \mathcal{V}_{J_{\mathcal{R}}} = \mathcal{V}_{\mathcal{R}}.
 \end{array}$$

**Proof.** The fact that  $\mathcal{R}_{\mathcal{V}}$  and  $\mathcal{V}_{\mathcal{R}}$  are intrinsic quantum relations and weak-\* closed  $\mathcal{M}'$ -bimodules is trivial. Points (v) and (vi) are the content of [Wea12, Theorem 2.32]. We shall prove only the rest of the points.

**Proof of (i).**  $\mathcal{V}_J$  is a weak-\* closed  $\mathcal{M}'$ -bimodule since it is the intersection of  $\{T \in \mathcal{B}(H) : \Phi_s(T) = 0\}$  for every  $s \in J$  and each of these subspaces is weak-\* closed and  $\mathcal{M}'$ -bimodular. It is also clear that  $\mathcal{V} \subset \mathcal{V}_{J_{\mathcal{V}}}$ , we only need to prove the converse. Let  $T \notin \mathcal{V}$ . Since  $\mathcal{V} \subset \mathcal{B}(H)$  is weak-\* closed there is, by the Hahn-Banach Theorem, a weak-\* continuous functional  $\phi : \mathcal{B}(H) \rightarrow \mathbb{C}$  such that  $\phi(S) = 0, \forall S \in \mathcal{V}$  but  $\phi(T) \neq 0$ . Any such functional is of the form  $\phi(A) = \langle \eta, (\mathbf{1} \otimes A)\xi \rangle$ , where  $\eta, \xi \in \ell_2 \otimes_2 H$ . Since  $\mathcal{V}$  is an  $\mathcal{M}'$ -bimodule we have that

$$\langle (\mathbf{1} \otimes x)\eta, (\mathbf{1} \otimes S)(\mathbf{1} \otimes y)\xi \rangle = 0,$$

where  $S \in \mathcal{V}$  and  $x, y \in \mathcal{M}'$ . Let  $P$  and  $Q$  be the orthogonal projections onto the subspaces of  $\ell_2 \otimes_2 H$  given by

$$H_1 = \overline{(\mathbf{1} \otimes \mathcal{M}')\eta}, \quad H_2 = \overline{(\mathbf{1} \otimes \mathcal{M}')\xi}.$$

These subspaces are  $(\mathbf{1} \otimes \mathcal{M}')$ -invariant, therefore  $P, Q \in (\mathbf{1} \otimes \mathcal{M})' = \mathcal{B}(\ell_2) \overline{\otimes} \mathcal{M}$ . Clearly we have  $P(\mathbf{1} \otimes \mathcal{V})Q = \{0\}$  but  $P(\mathbf{1} \otimes T)Q \neq 0$ . Let us write  $P = [p_{ij}]_{i,j}$  and  $Q = [q_{ij}]_{i,j}$ , where  $p_{ij}, q_{ij} \in \mathcal{M}$ . Notice that:

$$P(\mathbf{1} \otimes T)Q = [\Phi_{r_{ij}}(T)]_{ij},$$

where

$$r_{ij} = \sum_{k=1}^{\infty} p_{ik} \otimes q_{kj} \in \mathcal{M} \otimes_{eh} \mathcal{M}.$$

Since  $P(\mathbf{1} \otimes T)Q \neq 0$  there are  $i, j$  such that  $\Phi_{r_{ij}}(T) \neq 0$  but  $r_{ij} \in J_{\mathcal{V}}$  which implies that  $T \notin \mathcal{V}_{J_{\mathcal{V}}}$  and so  $\mathcal{V}_{J_{\mathcal{V}}} \subset \mathcal{V}$ , which concludes (i).

**Proof of (ii).** First, let us see that  $J_{\mathcal{V}}$  is  $\sigma(\mathcal{B}(H) \widehat{\otimes} S_1(H))$ -closed. Observe that  $\Phi_s(T) = 0$  iff  $\langle \xi, \Phi_s(T) \rangle = 0$  for every  $\xi \in \mathcal{B}(H)_*$ . Therefore

$$\{s \in \mathcal{M} \otimes_{eh} \mathcal{M} : \Phi_s(T) = 0\} = \bigcap_{\xi \in S_1(H)} \{s \in \mathcal{M} \otimes_{eh} \mathcal{M} : \langle \xi, \Phi_s(T) \rangle = 0\},$$

and so, the left hand side is pointwise weak-\* closed. Since the  $\sigma(\mathcal{B}(H) \widehat{\otimes} S_1(H))$  topology is finer than the pointwise weak-\* topology of  $\mathcal{CB}_{\mathcal{M}', \mathcal{M}'}^{\sigma}(\mathcal{B}(H))$  we have that  $\{s \in J : \Phi_s|_{\mathcal{V}} = 0\}$  is a weak-\* closed subspace. The fact that it is a left ideal follows trivially from the definition.

Let  $J$  be a  $\sigma(\mathcal{B}(H) \widehat{\otimes} S_1(H))$ -closed left ideal. Again, it is clear that  $J \subset J_{\mathcal{V}}$  we only have to prove the other containment. That is equivalent to prove that for every  $s_0 \notin J$  there is  $T \in \mathcal{B}(H)$  such that  $\Phi_s(T) = 0$  for every  $s \in J$  and  $\Phi_{s_0} \neq 0$ . By weak-\* closeness of  $J$  and the Hahn-Banach theorem there is a weak-\* continuous functional  $\phi \in (\mathcal{M} \otimes_{eh} \mathcal{M})_*$  such that  $\langle \phi, s \rangle = 0$  for every  $s \in J$  but  $\langle \phi, s_0 \rangle \neq 0$ . By Lemma 1.4 we have that

$$\langle \phi, s \rangle = \langle C, (\text{Id} \otimes \Phi_s)(B) \rangle,$$

where  $C \in S_1(\ell_2 \otimes_2 H)$  and  $B \in \mathcal{B}(\ell_2 \otimes_2 H)$ . We can decompose  $C = C_1 C_2^*$  where  $C_1, C_2 \in S_2(\ell_2 \otimes_2 H)$  and so

$$\langle \phi, s \rangle = \langle C_1, (\text{Id} \otimes \Phi_s)(B) C_2 \rangle$$

where  $(\text{Id} \otimes \Phi_s)(B) \in \mathcal{B}(\ell_2 \otimes_2 H)$  is acting on  $S_2(\ell_2 \otimes_2 H)$  by left multiplication and the duality pairing is that of  $S_2$  with itself. We have that, for every  $s \in J$ ,  $\langle \phi, s \rangle = 0$ , and so, since  $J$  is an ideal,  $\langle \phi, (x \otimes y) s \rangle = 0$ . Therefore

$$(2.1) \quad 0 = \langle (\mathbf{1} \otimes x^*) C_1, (\text{Id} \otimes \Phi_s)(B) (\mathbf{1} \otimes y) C_2 \rangle.$$

Let us define the closed subspaces  $H_1, H_2 \subset S_2(\ell_2 \otimes_2 H)$  given by

$$H_1 = \overline{(\mathbf{1} \otimes \mathcal{M}) C_1}, \quad H_2 = \overline{(\mathbf{1} \otimes \mathcal{M}) C_2}$$

and let  $P_i : S_2(\ell_2 \otimes_2 H) \rightarrow H_i$ , for  $i \in \{1, 2\}$ , be their orthogonal projections. We can identify isometrically  $S_2(\ell_2 \otimes_2 H) \cong \ell_2 \otimes_2 H \otimes_2 \ell_2 \otimes_2 H$ , such identification gives that  $\mathcal{B}(S_2(\ell_2 \otimes_2 H)) \cong \mathcal{B}(\ell_2) \overline{\otimes} \mathcal{B}(H) \overline{\otimes} \mathcal{B}(\ell_2) \overline{\otimes} \mathcal{B}(H)$ , where the first two components correspond to right multiplication and the other two correspond to left multiplication. Since  $H_1$  and  $H_2$  are  $\mathbb{C}\mathbf{1} \otimes \mathbb{C}\mathbf{1} \otimes \mathbb{C}\mathbf{1} \otimes \mathcal{M}$ -invariant the projections  $P_1, P_2$  belong to  $(\mathbb{C}\mathbf{1} \otimes \mathbb{C}\mathbf{1} \otimes \mathbb{C}\mathbf{1} \otimes \mathcal{M})' = \mathcal{B}(\ell_2) \overline{\otimes} \mathcal{B}(H) \overline{\otimes} \mathcal{B}(\ell_2) \overline{\otimes} \mathcal{M}'$ . Now, the identity 2.1 implies that

$$0 = P_1 (\text{Id} \otimes \Phi_s)(B) P_2,$$

where  $(\text{Id} \otimes \Phi_s)(B)$  is seen as an operator in  $\mathcal{B}(\ell_2) \overline{\otimes} \mathcal{B}(H) \otimes \mathbb{C}1 \otimes \mathbb{C}1$ . If  $s = \sum_k x_k \otimes y_k$  we have that

$$\begin{aligned} P_1 (\text{Id} \otimes \Phi_s)(B) P_2 &= \sum_k P_1 (\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes x_k) B (\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes y_k) P_2 \\ &= \sum_k (\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes x_k) (P_1 B P_2) (\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes y_k) \\ &= (\text{Id}_{\mathcal{B}(\ell_2 \otimes_2 H \otimes_2 \ell_2)} \otimes \Phi_s)(P_1 B P_2). \end{aligned}$$

Let  $T_\xi \in \mathcal{B}(H)$  be the operator given by  $(\xi \otimes \text{Id}_{\mathcal{B}(H)})(P_1 B P_1) \in \mathcal{B}(H)$ , where  $\xi \in \mathcal{B}(\ell_2 \otimes_2 H \otimes_2 \ell_2)_*$ . We have that  $\Phi_s(T_\xi) = 0$  for every  $s \in J$  since

$$\begin{aligned} \Phi_s(T_\xi) &= (\xi \otimes \Phi_s)(P_1 B P_1) \\ &= \langle \xi, (\text{Id}_{\mathcal{B}(\ell_2 \otimes_2 H \otimes_2 \ell_2)} \otimes \Phi_s)(P_1 B P_2) \rangle \\ &= 0. \end{aligned}$$

But there has to be a  $\xi_0 \in \mathcal{B}(\ell_2 \otimes_2 H \otimes_2 \ell_2)_*$  such that  $\Phi_{s_0}(T_{\xi_0}) \neq 0$ , otherwise

$$\langle \xi, P_1 (\text{Id}_{\mathcal{B}(\ell_2 \otimes_2 H \otimes_2 \ell_2)} \otimes \Phi_{s_0})(B) P_2 \rangle = 0,$$

for every  $\xi \in \mathcal{B}(\ell_2 \otimes_2 H \otimes_2 \ell_2)_*$  which implies that  $P_1 (\text{Id} \otimes \Phi_{s_0})(B) P_2 = 0$  but that is impossible since  $C_1$  and  $C_2$  are in the ranges of  $P_1$  and  $P_2$  respectively. The existence of such  $T_{\xi_0}$  finishes the proof.

**Proof of (iii).** We will start proving that  $\mathcal{R}_J$  is an intrinsic quantum relation. First, we have to see that  $\mathcal{R}_J$  is weak-\* open. Since  $J$  is  $\sigma(\mathcal{B}(H) \widehat{\otimes} S_1(H))$ -closed, so is  $\mathcal{B}(\ell_2) \overline{\otimes} J$ . The complementary  $(\mathcal{B}(\ell_2) \overline{\otimes} J)^c$  is weak-\* open and so is  $\mathcal{R}_J$ , since it is the reverse image of  $(\mathcal{B}(\ell_2) \overline{\otimes} J)^c$  under the function  $\odot : \mathcal{P}(\mathcal{B}(\ell_2) \overline{\otimes} \mathcal{M}) \times \mathcal{P}(\mathcal{B}(\ell_2) \overline{\otimes} \mathcal{M}) \rightarrow \mathcal{B}(\ell_2) \overline{\otimes} (\mathcal{M} \otimes_{eh} \mathcal{M})$ , which is weak-\* continuous by Lemma 2.2 (recall that over  $\mathcal{P}(\mathcal{B}(\ell_2) \overline{\otimes} \mathcal{M})$  the SOT, WOT,  $\sigma$ -SOT and  $\sigma$ -WOT coincide). Second, we are going to prove the properties (i), (ii), (ii) in Definition 1.1. It is trivial that  $(0, 0) \notin \mathcal{R}_J$ . For (ii) we have to prove that

$$\forall \alpha, \beta, (P_\alpha, Q_\beta) \in \mathcal{B}(\ell_2) \overline{\otimes} J \iff \left( \bigvee_\alpha P_\alpha, \bigvee_\beta Q_\beta \right) \in \mathcal{B}(\ell_2) \overline{\otimes} J.$$

For the implication  $(\implies)$  we use that if  $(P_\alpha, Q_\beta) \in \mathcal{B}(\ell_2) \overline{\otimes} J$  then

$$\left( \sum_\alpha P_\alpha, \sum_\beta Q_\beta \right) \in \mathcal{B}(\ell_2) \overline{\otimes} J,$$

but using that, for any family of projections  $(R_\gamma)_\gamma$

$$\left[ \sum_\gamma R_\gamma \right] = \bigvee_\gamma R_\gamma$$

and Lemma 2.2 (ii) we obtain that

$$\left( \bigvee_\alpha P_\alpha, \bigvee_\beta Q_\beta \right) = \left( \left[ \sum_\alpha P_\alpha \right], \left[ \sum_\beta Q_\beta \right] \right) \in \mathcal{B}(\ell_2) \overline{\otimes} J.$$

Proving  $(\impliedby)$  is clearly equivalent to proving that  $P \odot Q \in \mathcal{B}(\ell_2) \overline{\otimes} J$  implies that  $R \odot S \in \mathcal{B}(\ell_2) \overline{\otimes} J$  for any projections  $R \leq P$  and  $S \leq Q$ , but that follows trivially from Lemma 2.2 (i). For point (iv) we have that if  $P \odot [BQ] \in \mathcal{B}(\ell_2) \overline{\otimes} J$  then  $P \odot BQ \in \mathcal{B}(\ell_2) \overline{\otimes} J$  by Lemma 2.2. Since  $B \in \mathcal{B}(\ell_2) \otimes \mathbb{C}1$  we have that  $P \odot BQ =$

$PB \odot Q \in \mathcal{B}(\ell_2) \overline{\otimes} J$ , again by Lemma 2.2, that implies that  $[B^*P] \odot Q \in \mathcal{B}(\ell_2) \overline{\otimes} J$ . The other implication is proved similarly.

In order to prove the inclusion  $\mathcal{R}_{J_{\mathcal{R}}} \subset \mathcal{R}$  start by noticing that:

$$\begin{aligned} \mathcal{B}(\ell_2) \overline{\otimes} J_{\mathcal{R}} &= \mathcal{B}(\ell_2) \overline{\otimes} \overline{\{(\phi \otimes \text{Id})(X \odot Y) : \phi \in \mathcal{B}(\ell_2)_*, ([X^*], [Y]) \notin \mathcal{R}\}^{\text{w}^*}} \\ &= \overline{\text{span}^{\text{w}^*}\{X \odot Y : ([X^*], [Y]) \notin \mathcal{R}\}}. \end{aligned}$$

If we assume that  $P \odot Q \notin \overline{\text{span}^{\text{w}^*}\{X \odot Y : ([X^*], [Y]) \notin \mathcal{R}\}}$  then trivially we have that  $(P, Q) \in \mathcal{R}$ . For the other inclusion we shall use that, by (v),  $(P, Q) \in \mathcal{R}$  iff  $(P, Q) \in \mathcal{R}_{\mathcal{V}_{\mathcal{R}}}$  which happens only when  $P(\mathbf{1} \otimes A)Q \neq 0$  for some  $A \in \mathcal{V}_{\mathcal{R}}$ . Since the complete isometry  $\text{Id} \otimes \Phi : \mathcal{B}(\ell_2) \overline{\otimes} (\mathcal{M} \otimes_{eh} \mathcal{M}) \rightarrow \mathcal{CB}_{\mathcal{M}' \mathcal{M}'}^{\sigma}(\mathcal{B}(H), \mathcal{B}(\ell_2 \otimes_2 H))$  satisfies that  $(\text{Id} \otimes \Phi)_{X \odot Y}(A) = X(\mathbf{1} \otimes A)Y$  we have that

$$\begin{aligned} \overline{\text{span}^{\text{w}^*}\{X \odot Y : ([X^*], [Y]) \notin \mathcal{R}\}} &= \overline{\text{span}^{\text{w}^*}\{X \odot Y : [X^*](\mathbf{1} \otimes \mathcal{V}_{\mathcal{R}})[Y] = \{0\}\}} \\ &= \overline{\text{span}^{\text{w}^*}\{X \odot Y : (\text{Id} \otimes \Phi)_{[X^*] \odot [Y]}|_{\mathcal{V}_{\mathcal{R}}} = 0\}} \\ &\subset \{s : (\text{Id} \otimes \Phi)_s|_{\mathcal{V}_{\mathcal{R}}} = 0\}. \end{aligned}$$

But no pair  $(P, Q) \in \mathcal{R}$  satisfies that  $P \odot Q \in \{X \odot Y : ([X^*], [Y]) \notin \mathcal{R}\}$  since that will imply that  $(\text{Id} \otimes \Phi)_{P \odot Q}|_{\mathcal{V}_{\mathcal{R}}} = 0$  and that is a contradiction.

**Proof of (iv).** Let us see that  $J_{\mathcal{R}}$  is an ideal for every intrinsic quantum relation  $\mathcal{R}$ . To see that it is a linear subspace fix  $\phi_1, \phi_2 \in \mathcal{B}(\ell_2)_*$  and  $([X_1^*], [Y_1]) \notin \mathcal{R}$ ,  $([X_2^*], [Y_2]) \notin \mathcal{R}$ . If  $B_1 : \ell_2 \rightarrow \ell_2$  and  $B_2 : \ell_2 \rightarrow \ell_2$  are isometries whose ranges are orthogonal and complementary. We have that the operators

$$\begin{aligned} X &= (B_1 \otimes \mathbf{1})X_1(B_1^* \otimes \mathbf{1}) + (B_2 \otimes \mathbf{1})X_2(B_1^* \otimes \mathbf{1}) \\ Y &= (B_1 \otimes \mathbf{1})Y_1(B_1^* \otimes \mathbf{1}) + (B_2 \otimes \mathbf{1})Y_2(B_1^* \otimes \mathbf{1}) \end{aligned}$$

satisfy  $[X^*] = (B_1 \otimes \mathbf{1})[X_1^*](B_1^* \otimes \mathbf{1}) + (B_2 \otimes \mathbf{1})[X_2^*](B_1^* \otimes \mathbf{1})$ ,  $[Y] = (B_1 \otimes \mathbf{1})[Y_1](B_1^* \otimes \mathbf{1}) + (B_2 \otimes \mathbf{1})[Y_2](B_1^* \otimes \mathbf{1})$  and therefore, by [KW12, Lemma 2.29],  $([X^*], [Y]) \in \mathcal{R}$ . Now, a trivial calculation gives that

$$\begin{aligned} &(\phi_1 \otimes \text{Id})(X_1 \odot Y_1) + (\phi_2 \otimes \text{Id})(X_2 \odot Y_2) \\ &= ((B_1 \phi_1 B_1^* + B_2 \phi_2 B_2^*) \otimes \text{Id})(X \odot Y), \end{aligned}$$

where  $B_i \phi_j B_i^*(x) = \phi_j(B_i^* x B_i)$ . The fact that  $J_{\mathcal{R}}$  is closed by scalar multiplication is trivial. It is also  $\sigma(\mathcal{B}(H) \otimes S_1(H))$ -closed by construction. It only rest to see that it is absorbent for the multiplication. It is enough to prove that  $(z \otimes t)J_{\mathcal{R}} \subset J_{\mathcal{R}}$  for every  $z, t \in \mathcal{M}$ . We have that

$$(z \otimes t)(\phi \otimes \text{Id})(X \odot Y) = (\phi \otimes \text{Id})((\mathbf{1} \otimes z)X \odot Y(\mathbf{1} \otimes t)).$$

Now, using that  $[Y(\mathbf{1} \otimes t)] \leq [Y]$  and  $[X^*(\mathbf{1} \otimes z^*)] \leq [X^*]$  and applying point (ii) in Definition 1.1 gives the desired result.

The inclusion  $J_{\mathcal{R}_J} \subset J$  is easy to prove. Recall that if  $s \in \mathcal{B}(\ell_2) \overline{\otimes} J$  then  $(\phi \otimes \text{Id})(s) \in J$ . Using that together with Lemma 2.2(ii) gives

$$\begin{aligned} &\overline{\{(\phi \otimes \text{Id})(X \odot Y) : ([X^*], [Y]) \notin \mathcal{R}_J, \phi \in \mathcal{B}(\ell_2)_*\}^{\text{w}^*}} \\ &= \overline{\{(\phi \otimes \text{Id})(X \odot Y) : [X^*] \odot [Y] \in \mathcal{B}(\ell_2) \overline{\otimes} J, \phi \in \mathcal{B}(\ell_2)_*\}^{\text{w}^*}} \\ &\subset J. \end{aligned}$$

For the reciprocal inclusion  $J \subset J_{\mathcal{R}_J}$  we need to see that if  $s \in J$  then there are  $\phi \in \mathcal{B}(\ell_2)_*$ ,  $X, Y \in \mathcal{B}(\ell_2) \overline{\otimes} \mathcal{M}$  with  $([X^*], [Y]) \notin \mathcal{R}_J$  such that  $s = (\phi \otimes \text{Id})(X \odot Y)$ .

Note that we can express

$$s = \sum_{k=0}^{\infty} x_k \otimes y_k = (\omega_{e_1, e_1} \otimes \text{Id})(X \odot Y),$$

as

$$X = \sum_{k=0}^{\infty} x_k \otimes e_{1k} \in \mathcal{B}(\ell_2) \overline{\otimes} \mathcal{M} \quad \text{and} \quad Y = \sum_{k=0}^{\infty} y_k \otimes e_{k1} \in \mathcal{B}(\ell_2) \overline{\otimes} \mathcal{M}.$$

We only have to prove that  $([X^*], [Y]) \notin \mathcal{R}_J$ , i.e. that  $[X^*] \odot [Y] \in \mathcal{B}(\ell_2) \overline{\otimes} J$ . Again, by Lemma 2.2(ii), we only have to see that  $X \odot Y \in \mathcal{B}(\ell_2) \overline{\otimes} J$ , which is equivalent to see that for every  $\phi \in \mathcal{B}(\ell_2)_*$ ,  $(\phi \otimes \text{Id})(X \odot Y) \in J$ . Notice that if  $P$  is the projection on the 1-dimensional subspace spanned by  $e_1$ , then  $X \odot Y = (P \otimes \mathbf{1})(X \odot Y)(P \otimes \mathbf{1})$ . Therefore  $(\phi \otimes \text{Id})(X \odot Y) = (P \phi P \otimes \text{Id})(X \odot Y) = (\lambda \omega_{e_1, e_1} \otimes \text{Id})(X \odot Y) = \lambda s \in J$ , for some  $\lambda \in \mathbb{C}$ . That finishes the proof of (iv).

Since we have already proved (i)-(vi) we have that (4)-(6) can be deduced from (1)-(3). We will prove only those first three cases, which are easy after the previous results.

**Proof of (1).** We have that

$$\mathcal{R}_{J_{\mathcal{V}}} = \{(P, Q) \in \mathcal{P}(\mathcal{B}(\ell_2) \overline{\otimes} \mathcal{M})^2 : P \odot Q \notin \mathcal{B}(\ell_2) \overline{\otimes} J_{\mathcal{V}}\}$$

and that

$$\mathcal{B}(\ell_2) \overline{\otimes} J_{\mathcal{V}} = \{s \in \mathcal{B}(\ell_2) \overline{\otimes} (\mathcal{M} \otimes_{eh} \mathcal{M}) : (\text{Id} \otimes \Phi)_s|_{\mathcal{V}} = 0\},$$

therefore

$$\begin{aligned} \mathcal{R}_{J_{\mathcal{V}}} &= \{(P, Q) \in \mathcal{P}(\mathcal{B}(\ell_2) \overline{\otimes} \mathcal{M})^2 : (\text{Id} \otimes \Phi)_{P \odot Q}|_{\mathcal{V}} \neq 0\} \\ &= \{(P, Q) \in \mathcal{P}(\mathcal{B}(\ell_2) \overline{\otimes} \mathcal{M})^2 : P(\mathbf{1} \otimes \mathcal{V})Q \neq \{0\}\} = \mathcal{R}_{\mathcal{V}}. \end{aligned}$$

**Proof of (2).** Let us start by seeing that  $J_{\mathcal{R}} \subset J_{\mathcal{V}_{\mathcal{R}}}$ . If  $z = (\phi \otimes \text{Id})(X \odot Y)$ , with  $([X^*], [Y]) \notin \mathcal{R}$ , then  $(\text{Id} \otimes \Phi)_z(T) = (\phi \otimes \text{Id})(X(\mathbf{1} \otimes T)Y) = (\phi \otimes \text{Id})(X[X^*](\mathbf{1} \otimes T)[Y^*]Y)$ . So if  $T \in \mathcal{V}_{\mathcal{R}}$  then  $(\text{Id} \otimes \Phi)_z(T) = 0$ . For the converse inclusion let  $s \in J_{\mathcal{V}_{\mathcal{R}}}$  and express  $s$  as  $s = (\omega_{e_1, e_1} \otimes \text{Id})(X \odot Y)$  like in the proof of (iv). We have that  $X(\mathbf{1} \otimes \mathcal{V}_{\mathcal{R}})Y = 0$  and so  $[X^*](\mathbf{1} \otimes \mathcal{V}_{\mathcal{R}})[Y] = 0$  which implies that  $([X^*], [Y]) \notin \mathcal{R}_{\mathcal{V}_{\mathcal{R}}} = \mathcal{R}$  and so  $s \in J_{\mathcal{R}}$ .

**Proof of (3).** The inclusion  $\mathcal{V}_J \subset \mathcal{V}_{\mathcal{R}_J}$  is trivial. In order to prove the converse,  $\mathcal{V}_{\mathcal{R}_J} \subset \mathcal{V}_J$ , fix  $S \in \mathcal{V}_{\mathcal{R}_J}$ . We have that  $X(\mathbf{1} \otimes S)Y = 0$ ,  $\forall ([X^*], [Y])$  such that  $[X^*] \odot [Y] \in \mathcal{B}(\ell_2) \overline{\otimes} J$ . Then, for any  $\phi \in \mathcal{B}(\ell_2)_*$  we have that

$$0 = (\phi \otimes \text{Id})(X(\mathbf{1} \otimes S)Y) = (\phi \otimes \text{Id})((\text{Id} \otimes \Phi)_{X \odot Y}(S)) = \Phi_{(\phi \otimes \text{Id})(X \odot Y)}(S).$$

Therefore  $\Phi_z(S) = 0$  for every  $z \in J_{\mathcal{R}_J} = J$ .  $\square$

Recall that the technique of the proof of point (i) follows exactly the same lines of [Wea12, Lemma 2.8].

**Remark 2.1.** Observe that, a priori, it is not clear why all  $\sigma(\mathcal{B}(H) \widehat{\otimes} S_1(H))$ -closed ideals are closed in the coarser pointwise weak-\* topology. Such result is obtained as a consequence from Theorem 2.3.(ii).

### 3. Invariant Quantum Relations

Let  $\mathcal{A}$  be a von Neumann algebraic *quantum group* with comultiplication  $\Delta$ , see [VD14] for a precise definition, we will say that  $\mathcal{M}$  is a *quantum homogeneous space* if there is a normal,  $*$ -homomorphism  $\sigma : \mathcal{M} \rightarrow \mathcal{A} \overline{\otimes} \mathcal{M}$ , called the *coaction*, satisfying the natural coassociativity identity

$$(\text{Id} \otimes \sigma) \sigma = (\Delta \otimes \text{Id}) \sigma$$

If  $\mathcal{M} \subset \mathcal{B}(H)$  is an *standard form* for the von Neumann algebra  $\mathcal{M}$ , we have that, after endowing  $H$  with its row (resp. column) operator space structure, the coaction  $\sigma$  extends to a complete isometry  $\sigma_2 : H^r \rightarrow \mathcal{A} \overline{\otimes} H^r$ . We will say that an operator  $T \in \mathcal{B}(H)$  is  $\sigma$ -*equivariant* iff

$$\sigma_2 T = (\text{Id} \otimes T) \sigma_2$$

and we will denote by  $\mathcal{B}(H)^\sigma$  the space of  $\sigma$ -equivariant operators. Similarly, we say that a quantum relation  $\mathcal{V}$  over  $\mathcal{M}$  is  $\sigma$ -*invariant*, or simply *invariant* if the coaction is understood from the context, iff it is generated (as an operator  $\mathcal{M}'$ -bimodule) by  $\sigma$ -equivariant operators. If  $\mathcal{V}$  is generated by equivariant operators, then it is generated by the equivariant operators inside  $\mathcal{V}$ , therefore  $\mathcal{V}$  is invariant iff

$$\mathcal{V} =_{\mathcal{M}'} \langle \mathcal{V} \cap \mathcal{B}(H)^\sigma \rangle_{\mathcal{M}'} = \overline{\text{span}^{\text{w}^*}} \{xTy : x, y \in \mathcal{M}', \quad T \in \mathcal{V} \cap \mathcal{B}(H)^\sigma\}.$$

From now on we will denote  $\mathcal{V} \cap \mathcal{B}(H)^\sigma$  by  $\mathcal{V}^\sigma$ . Our purpose in this section is to study invariant quantum relations. Interesting examples of quantum homogeneous spaces include, among others, the ones listed below.

**Classical homogeneous spaces:** Let  $G$  is a locally compact Hausdorff group and  $X$  be a measurable  $G$ -space.  $L_\infty(G)$  is clearly a quantum group with the comultiplication given by  $\Delta(f)(g, h) = f(gh)$ . Similarly, we can define the coaction  $\sigma : L_\infty(X) \rightarrow L_\infty(G) \overline{\otimes} L_\infty(X)$  given by  $\sigma(f)(g, x) = f(gx)$ . To solidify our intuition let us see what happens when  $X$  is discrete. In that case quantum relations over  $L_\infty(X)$  are just subsets  $R \subset X \times X$ . Recall that a classical relation  $R \subset X \times X$  is  $G$ -invariant iff

$$(3.1) \quad (x, y) \in R \iff (gx, gy) \in R, \quad \forall g \in G.$$

We are going to see that such relations correspond with  $\sigma$ -invariant quantum relations. An operator  $T = [a_{xy}]_{x, y \in X} \in \mathcal{B}(L_2 X)$  is  $\sigma$ -equivariant iff it commutes with the action  $\sigma_g(f)(x) = f(g^{-1}x)$ , therefore the set

$$R_T = \{(x, y) \in X \times X : \langle e_x, Te_y \rangle \neq 0\} \subset X \times X$$

satisfies (3.1) and the same goes for  $R_{\mathcal{V}}$ , where  $\mathcal{V} =_{\mathcal{R}G} \langle \mathcal{V}^\sigma \rangle_{\mathcal{R}G}$ , since

$$R_{\mathcal{V}} = \bigcup_{T \in \mathcal{V}^\sigma} \{(x, y) \in X \times X : \langle e_x, Te_y \rangle \neq 0\}.$$

This proves that any  $\sigma$ -invariant quantum relation over a discrete space  $X$  corresponds to an invariant relation  $R \subset X \times X$ . The reciprocal is shown similarly.

**Group von Neumann algebras:** Let  $G$  be a locally compact Hausdorff group,  $L_2(G)$  the  $L_2$ -space with respect to the left Haar measure and  $\lambda : G \rightarrow \mathcal{U}(L_2 G)$  be the unitary representation given by  $\lambda_g(\xi)(h) = \xi(g^{-1}h)$ , where  $\xi \in L_2(G)$ . The (left) group von Neumann algebra  $\mathcal{L}G$  is given by

$$\mathcal{L}G = \{\lambda_g : g \in G\}'' \subset \mathcal{B}(L_2 G).$$

The natural comultiplication structure  $\Delta : \mathcal{L}G \rightarrow \mathcal{L}G \otimes_{eh} \mathcal{L}G$  is given by  $\lambda_g \mapsto \lambda_g \otimes \lambda_g$ . In this case the commutant  $\mathcal{L}G'$  is given by the (right) group von Neumann algebra

$$\mathcal{R}G = \{\rho_g : g \in G\}'' \subset \mathcal{B}(L_2G)$$

where  $\rho_g(\xi)(h) = \xi(hg) \Delta(g)^{\frac{1}{2}}$ , where  $\xi \in L_2(G)$  is right regular representation. We can consider  $\mathcal{L}G$  a quantum homogeneous space over itself with the multiplication as coaction. The representation  $\mathcal{L}G \subset \mathcal{B}(L_2G)$  is standard, and the  $\Delta$ -equivariant operators are given by the subalgebra  $L_\infty(G)$  acting by multiplication operation. It is also illustrative to observe that if we take the GNS representation associated with the canonical Plancherel weight  $\varphi$ , see [Ped79],  $\mathcal{L}G \subset \mathcal{B}(L_2(\mathcal{L}G, \varphi))$ , then an element  $T : L_2(\mathcal{L}G, \varphi) \rightarrow L_2(\mathcal{L}G, \varphi)$  is  $\Delta$ -equivariant iff it is a noncommutative Fourier multiplier over  $L_2(\mathcal{L}G)$ , in the sense of [CdLS15, 3.7]. By the Plancherel theorem, the algebra of such multipliers is equivalent to  $L_\infty(G)$ .

**Quantum Torii:** One family of von Neumann algebras that has received a considerable amount of attention is that of *quantum torii*  $\mathcal{A}_\theta^n \subset L_2(\mathbb{T}^n)$ . In such case the coaction is given by  $\sigma : \mathcal{A}_\theta^n \rightarrow L_\infty(\mathbb{T}^n) \overline{\otimes} \mathcal{A}_\theta^n$ . Quantum relations on quantum torii have been considered before in [Wea12, Section 2.7].

Here, we will mainly focus our attention on the case of  $\mathcal{M} = \mathcal{L}G$ . Our purpose is to describe the ideals associated with invariant quantum relations over  $\mathcal{L}G$ . For that, we need to recall some results on the representation of completely bounded  $\mathcal{R}G$ -bimodular operators preserving the  $\Delta$ -equivariant operators. Let  $MG$  be the Banach algebra of finite measures with the o.s.s. given by  $C_0(G)^* = MG$ . Apart from the weak-\* topology given by  $\sigma(C_0G)$  in  $MG$  we have the strictly finer topology generated by evaluation against every bounded continuous function  $\sigma(C_bG)$ . Reasoning like before, since  $MG$  is  $\sigma(C_bG)$ -closed, the  $\sigma(C_bG)$  topology induces another predual for  $MG$ . The subalgebra of point measures  $\ell_1(G) \subset MG$  is of course  $\sigma(C_bG)$ -dense. We define a multiplicative and injective map  $j : \ell_1(G) \rightarrow \mathcal{L}G \otimes_{eh} \mathcal{L}G$  by  $\delta_g \mapsto \lambda_g \otimes \lambda_{g^{-1}}$ . The following theorem assure that there is an injective and weak-\* continuous extension to  $MG$  and characterizes its range as normal  $\mathcal{R}G$ -bimodular, c.b. maps preserving  $\mathcal{B}(L_2G)^\Delta = L_\infty(G)$ . We will denote the algebra of such operators by  $\mathcal{CB}_{\mathcal{R}G-\mathcal{R}G}^{\sigma, L_\infty(G)}(\mathcal{B}(L_2G)) \subset \mathcal{CB}_{\mathcal{R}G-\mathcal{R}G}^\sigma(\mathcal{B}(L_2G))$ . Such algebra is closed, with respect to the natural weak-\* topologies of  $\mathcal{CB}_{\mathcal{R}G-\mathcal{R}G}^\sigma(\mathcal{B}(L_2G))$  and so it inherits both the  $\sigma(\mathcal{B}(L_2G) \widehat{\otimes} S_1(L_2G))$  and the  $\sigma(\mathcal{K}(L_2G) \widehat{\otimes} S_1(L_2G))$  topologies.

**Theorem 3.1.** ([NRS08, Theorem 3.2]) *Let  $G$  be a locally compact group. There is a  $\sigma(C_b)$  to  $\sigma(\mathcal{B} \widehat{\otimes} S_1)$  continuous, multiplicative and injective complete isometry  $j : MG \rightarrow \mathcal{L}G \otimes_{eh} \mathcal{L}G$  extending the map  $\delta_g \mapsto \lambda_g \otimes \lambda_{g^{-1}}$ . Furthermore, the following diagram commute*

$$\begin{array}{ccc} MG & \xrightarrow{j} & \mathcal{L}G \otimes_{eh} \mathcal{L}G \\ \downarrow \Theta & & \downarrow \Phi \\ \mathcal{CB}_{\mathcal{R}G-\mathcal{R}G}^{\sigma, L_\infty(G)}(\mathcal{B}(H)) & \xrightarrow{\subseteq} & \mathcal{CB}_{\mathcal{R}G-\mathcal{R}G}^\sigma(\mathcal{B}(H)) \end{array}$$

*In particular,  $\Theta : MG \rightarrow \mathcal{CB}_{\mathcal{R}G-\mathcal{R}G}^\sigma(\mathcal{B}(L_2G))$  is a complete isometry whose range is  $\mathcal{CB}_{\mathcal{R}G-\mathcal{R}G}^{\sigma, L_\infty(G)}(\mathcal{B}(L_2G))$ . The topology induced by  $\sigma(\mathcal{K} \widehat{\otimes} S_1)$  in  $MG$  is the just  $\sigma(C_0)$ , while the topology induced by  $\sigma(\mathcal{B} \widehat{\otimes} S_1)$  is  $\sigma(C_b)$ .*



We will, perhaps ambiguously, denote by  $\Theta$  either the map  $\Theta : MG \rightarrow \mathcal{CB}_{\mathcal{RG}-\mathcal{RG}}^\sigma(\mathcal{B}(L_2G))$  or the restriction to its image.

We will briefly sketch the proof of the theorem above since some of its ideas will be used in the forthcoming results. But, before that, we need to recall a few well known facts on the theory of crossed products. Let  $r : G \rightarrow \text{Aut}(L_\infty G)$  be the normal right-translation action given by  $r_g(f)(x) = f(xg)$  noticing that by the Takai-Takesaki duality theorem, see [Tak73], we have

$$L_\infty(G) \rtimes_r G = \mathcal{B}(L_2G),$$

where  $\rtimes$  is notation for the (weak-\* closed) spatial crossed product. The action  $r$  is spatially implemented on  $L_\infty(G) \subset \mathcal{B}(L_2G)$  by the right regular representation, i.e.  $r_g(f) = \rho_g f \rho_{g^{-1}}$  and so we obtain that

$$L_\infty(G) \rtimes_r G = \{L_\infty(G), \mathcal{RG}\}'' = \mathcal{B}(L_2G).$$

As a consequence, we can identify  $L_\infty(G) \rtimes \mathbf{1} \subset L_\infty(G) \rtimes_r G = \mathcal{B}(L_2G)$  with the algebra of  $\Delta$ -equivariant operators.

**Proof.** First, we are going to see that the map  $\Theta : MG \rightarrow \mathcal{CB}_{\mathcal{RG}-\mathcal{RG}}^\sigma(\mathcal{B}(L_2G))$  is surjective. Observe that  $\Theta_\mu$  acts on  $L_\infty(G) \subset \mathcal{B}(L_2G)$  by left convolution, i.e.  $\Theta_\mu(f) = \mu * f$ . Notice also that, if  $\Psi : \mathcal{B}(L_2G) \rightarrow \mathcal{B}(L_2G)$  is a normal and  $\mathcal{RG}$ -bimodular map, its restriction  $\Psi|_{L_\infty(G)} : L_\infty(G) \rightarrow L_\infty(G)$  determines the map  $\Psi$  since  $L_\infty(G)$  and  $\mathcal{RG}$  generate the whole von Neumann algebra  $\mathcal{B}(L_2G)$  by the Takai-Takesaki theorem. Furthermore, since  $\Psi$  preserves  $L_\infty(G)$ , we have that, for every  $f \in L_\infty(G)$

$$\rho_g \Psi(f) = \Psi(\rho_g f) = \Psi(\rho_g f \rho_{g^{-1}} \rho_g) = \Psi(r_g f) \rho_g$$

and so  $\Psi|_{L_\infty(G)}$  is a right-translation equivariant operator, i.e.  $r_g \Psi = \Psi r_g$ . But then, any such operator is actually given by left convolution with a finite measure, see [Wen52]. So  $\Psi(f) = \mu * f = \Theta_\mu(f)$  and since  $\Psi$  and  $\Theta$  coincide in  $L_\infty(G)$  they are equal.

Reciprocally, if we pick a measure  $\mu \in MG$  we have that the map  $T_\mu : L_\infty(G) \rightarrow L_\infty(G)$  given by  $f \mapsto \mu * f$  is a normal bounded operator commuting with  $r_g$  for all  $g \in G$ . Since  $L_\infty(G)$  is an abelian operator space we have that  $T_\mu$  is c.b. and that

$$\|T_\mu\|_{\text{cb}} = \|\mu\|_{MG}.$$

But for any crossed product there is a normal injective \*-homomorphism  $\iota : L_\infty(G) \rtimes G \rightarrow L_\infty(G) \overline{\otimes} \mathcal{B}(L_2G) \subset \mathcal{B}(L_2G \otimes_2 L_2G)$ . To define such embedding  $\iota$  fix an element  $\xi \in L_2(G \times G)$ . The action of  $\iota(f \rho_{g_0})$  on  $\xi$  is given by

$$j(f \rho_{g_0}) \xi(g, h) = f(g h^{-1} g_0^{-1}) \xi(g, g_0^{-1} h).$$

while for general  $x \in L_\infty(G) \rtimes_r G$  we just extend linearly and take weak-\* limits. Such embedding appears naturally in the crossed product construction. It satisfies that, if  $T$  is equivariant, then

$$\begin{array}{ccc} L_\infty(G) \rtimes G & \xhookrightarrow{\quad \iota \quad} & L_\infty(G) \overline{\otimes} \mathcal{B}(L_2G) \\ \downarrow T \rtimes \text{Id} & & \downarrow T \otimes \text{Id} \\ L_\infty(G) \rtimes G & \xhookrightarrow{\quad \iota \quad} & L_\infty(G) \overline{\otimes} \mathcal{B}(L_2G) \end{array}$$

As a consequence, if  $T$  is completely bounded so is  $T \rtimes \text{Id}$  and

$$\|T \rtimes \text{Id}\|_{\text{cb}} = \|T \otimes \text{Id}\|_{\text{cb}}$$

After identifying  $L_\infty(G) \rtimes_r G$  with  $\mathcal{B}(L_2G)$ , we get that  $T_\mu \rtimes \text{Id}$  is the only normal and  $\mathcal{R}G$ -bimodular extension of  $T_\mu$ . Therefore  $\Theta_\mu = T_\mu \rtimes \text{Id}$  and so  $\Theta$  is well defined and isometric. Since  $\Theta$  clearly factors through  $\mathcal{L}G \otimes_{eh} \mathcal{L}G$  we also obtain that  $j$  is a complete isometry.  $\square$

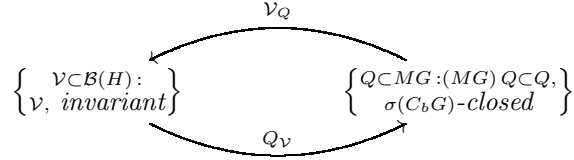
The result above goes back to Wendel [Wen52] but the formulation is taken from [NRS08], whose main contribution is to generalize the result from  $\mathcal{L}G$  to its quantum group dual  $L_\infty(G)$  obtaining a complete isomorphism  $\hat{\Theta} : M_{\text{cb}}AG \rightarrow \mathcal{CB}_{L_\infty(G)-L_\infty(G)}^{\sigma, \mathcal{L}G}(\mathcal{B}(L_2G))$ , where  $M_{\text{cb}}AG$  is the space of completely bounded multipliers of the Fourier algebra  $AG$ . It is also worth pointing out that both results can be unified using the language of quantum groups, see [JNR09].

The main result of this section is that invariant quantum relations over  $\mathcal{L}G$  are in bijective correspondence with weak-\* closed left ideals inside  $MG$ .

**Theorem 3.2.** *Let  $\mathcal{L}G \subset \mathcal{B}(H)$  be as above. If  $\mathcal{V}$  is an invariant quantum relation over  $\mathcal{L}G$  and  $Q \subset MG$  is a  $\sigma(C_bG)$ -closed left ideal, then the following maps*

- (1)  $Q_{\mathcal{V}} = \{\mu \in MG : \Theta_\mu|_{\mathcal{V}} = 0\},$
- (2)  $\mathcal{V}_Q = \{T \in \mathcal{B}(H) : \Theta_\mu(T) = 0, \quad \forall \mu \in Q\},$

*are bijective and inverse of each other, see diagram below.*



Let us denote by  $V_Q^\Delta$  the set

$$\mathcal{V}_Q^\Delta = \{T \in \mathcal{B}(H)^\Delta : \Theta_\mu(T) = 0, \quad \forall \mu \in Q\}.$$

The proof of Theorem 3.2 requires the following two lemmas.

**Lemma 3.3.** *If  $Q \subset MG$  is a  $\sigma(C_bG)$ -closed left ideal, then*

$${}_{\mathcal{R}G} \langle \mathcal{V}_Q^\Delta \rangle_{\mathcal{R}G} = \mathcal{V}_Q.$$

**Proof.** After identifying  $\mathcal{B}(L_2G)$  with  $L_\infty(G) \rtimes_r G$  again, we have that  $\Theta_\mu = T_\mu \rtimes \text{Id}$ , where  $T_\mu$  is the left convolution operator associated to  $\mu$ . We have that

$$\mathcal{V}_Q = \bigcap_{\mu \in Q} \ker(T_\mu \rtimes \text{Id}), \quad \mathcal{V}_Q^\Delta = \bigcap_{\mu \in Q} \ker(T_\mu).$$

But now we use that if  $T$  is a  $r$ -equivariant operator then

$$\begin{aligned} \ker(T \rtimes \text{Id}) &= \overline{\text{span}^{\text{w}^*}(\ker(T) \mathcal{R}G)} \\ &= \overline{\text{span}^{\text{w}^*}\{f \rho_g : f \in \ker(T), g \in G\}} \\ &\subset {}_{\mathcal{R}G} \langle \ker(T) \rangle_{\mathcal{R}G}. \end{aligned}$$

Using that the intersection of closures is larger than the closure of the intersections we get that

$$\begin{aligned} \mathcal{V}_Q &= \bigcap_{\mu \in Q} \ker(T_\mu \rtimes \text{Id}) \\ &\subset \bigcap_{\mu \in Q} {}_{\mathcal{R}G} \langle \ker(T_\mu) \rangle_{\mathcal{R}G} \\ &\subset {}_{\mathcal{R}G} \left\langle \bigcap_{\mu \in Q} \ker(T_\mu) \right\rangle_{\mathcal{R}G} = {}_{\mathcal{R}G} \langle \mathcal{V}_Q^\Delta \rangle_{\mathcal{R}G}. \end{aligned}$$

The other inclusion is trivial since  $\mathcal{V}_Q^\Delta \subset \mathcal{V}_Q$  and  $\mathcal{V}_Q$  is a  $\mathcal{R}G$ -bimodule.  $\square$

**Lemma 3.4.** *Let  $Q \mapsto \mathcal{V}_Q^\Delta$  and  $\mathcal{V}^\Delta \mapsto Q_{\mathcal{V}^\Delta}$  be as above, we have that*

- (1)  $Q_{\mathcal{V}_Q^\Delta} = Q$ .
- (2)  $\mathcal{V}^\Delta_{Q_{\mathcal{V}^\Delta}} = \mathcal{V}^\Delta$ .

**Proof.** Let us start by (1). It is trivial that  $Q \subset Q_{\mathcal{V}_Q}$ . We only have to prove the reverse inclusion. Assume that  $Q_{\mathcal{V}_Q}$  is greater than  $Q$ . Then by the Hahn-Banach Theorem, for any  $\mu_0 \in Q_{\mathcal{V}_Q} - Q$  we can take a functional  $f_0 \in C_b(G)$  such that  $\langle \mu_0, f_0 \rangle \neq 0$  but  $\langle \mu, f_0 \rangle = 0$ , for every  $\mu \in Q$ . Since  $Q$  is a translation invariant space we have that  $\mu * f_0 = 0$  for every  $\mu \in Q$  but  $\mu_0 * f_0 \neq 0$ . The first condition implies that  $f_0 \in \mathcal{V}_Q$ , which contradicts the fact that  $\mu_0 \in Q_{\mathcal{V}_Q}$ .

For (2) it is again clear that  $\mathcal{V}^\Delta \subset \mathcal{V}_{Q_{\mathcal{V}^\Delta}}$  and we only have to prove the converse inclusion. By similar means using the Hahn-Banach theorem and the translation invariance of  $\mathcal{V}^\Delta$  we get the result.  $\square$

Now, we can proceed to prove the main correspondence theorem.

**Proof.(of Theorem 3.2)** Let us start seeing that  $Q_{\mathcal{V}_Q}$  is a  $\sigma(C_b G)$ -closed ideal. Notice that,  $\mu * f = 0$  if and only if  $\langle g, \mu * f \rangle = 0$  for every  $g \in L_1(G)$ , but  $\langle g, \mu * f \rangle = \langle \mu, \tilde{f} * g \rangle$ , where  $\tilde{f}(x) = f(x^{-1})$ . Since  $\tilde{f} * g$  is a right uniformly bounded function in  $C_b(G)$ , the kernel of  $\mu \mapsto \mu * f$  is  $\sigma(C_b G)$ -closed and so is  $Q_{\mathcal{V}_Q}$ . The fact that  $\mathcal{V}_Q$  is a weak-\* closed  $\mathcal{R}G$ -bimodule is immediate since  $\Theta_\mu$  is weak-\* continuous  $\mathcal{R}G$ -bimodular map. The fact that is  $\Delta$ -invariant follows from 3.3. To prove that  $Q = Q_{\mathcal{V}_Q}$  we just apply the following lemmas.

$$\begin{aligned} Q &= Q_{\mathcal{V}_Q^\Delta} && \text{(by Lemma 3.4)} \\ &= {}_{\mathcal{R}G} \langle \mathcal{V}_Q^\Delta \rangle_{\mathcal{R}G} \\ &= Q_{\mathcal{V}_Q} && \text{(by Lemma 3.3).} \end{aligned}$$

Similarly, taking the  $\mathcal{R}G$ -bimodules generated by the left and the right hand side of (2), we get that

$${}_{\mathcal{R}G} \langle \mathcal{V}^\Delta \rangle_{\mathcal{R}G} = {}_{\mathcal{R}G} \langle \mathcal{V}_{Q_{\mathcal{V}^\Delta}}^\Delta \rangle_{\mathcal{R}G} = \mathcal{V}_{Q_{\mathcal{V}^\Delta}} = \mathcal{V}_{Q_{\mathcal{R}G} \langle \mathcal{V}^\Delta \rangle_{\mathcal{R}G}}.$$

But, by  $\Delta$ -invariance, the leftmost element is  $\mathcal{V}$  and the rightmost is  $\mathcal{V}_{Q_{\mathcal{V}}}$  and we conclude.  $\square$

**Remark 3.1.** We have exposed here the theory of invariant quantum relations for  $\mathcal{L}G$ . The same proof above works, after [NRS08] and [JNR09], for a general quantum group  $(\mathcal{A}, \Delta)$  just by replacing left ideals in  $MG$  by left ideals in  $M_{\text{cb}}A(\mathcal{A})$ .

Recall that if  $G = \mathbb{Z}^n$ , or any other abelian discrete group, then  $\mathcal{L}G = L_\infty(\mathbb{T}^n)$  and any ideal  $Q$  of  $MG = \ell_1(\mathbb{Z}^n)$  correspond to a closed subset  $C_Q \subset \hat{G}$  and such correspondence is injective. Nevertheless, not every ideal in  $MG$  is  $\sigma(C_b G)$ -closed and therefore not all closed subsets will appear in the image of the correspondence. We have that, in the invariant case, any quantum relation  $\mathcal{V}$  over  $L_\infty(\mathbb{T}^n)$  is actually a topological relation given by

$$(\theta_1, \theta_2) \in R \iff \theta_1^{-1} \theta_2 \in C,$$

where  $C \subset \mathbb{T}^n$  is a closed set.

#### 4. Remarks on $L_p$ - $L_q$ versions of Quantum Relations

In the introduction of [Wea12] it is stated that the natural, albeit naive, candidate for quantized relations over a von Neumann algebra  $\mathcal{M}$  are the projections on  $\mathcal{M} \overline{\otimes} \mathcal{M}_{\text{op}}$ , but that such objects do not have desirable properties. The question of which properties are missed is left unanswered there. Our aim here is to give an intuitive explanation on why there is no well-behaved composition operation between projection in  $\mathcal{M} \overline{\otimes} \mathcal{M}_{\text{op}}$ . After that, we will see that there is a larger family of *generalized quantum relations* that contains both quantum relations and projections in  $\mathcal{M} \overline{\otimes} \mathcal{M}_{\text{op}}$  as particular cases.

Start recalling that if  $\mathcal{M}$  is a von Neumann algebra and  $\phi$  is a normal, faithful and semifinite weight we can define the noncommutative  $L_p$ -spaces  $L_p(\mathcal{M}, \phi)$ , or simply  $L_p(\mathcal{M})$  if  $\phi$  is understood from the context, see [PX03]. Such spaces generalize the classical  $L_p$ -spaces  $L_p(X, \mu)$  when  $\mathcal{M} = L_\infty(X)$  and there are isomorphic identifications of  $L_\infty(\mathcal{M})$  with  $\mathcal{M}$ , of  $L_2(\mathcal{M})$  with the GNS construction of  $\phi$  and of  $L_1(\mathcal{M})$  with the predual  $\mathcal{M}_*$ . There is a canonical o.s.s. for these spaces given by operator space interpolation between

$$\begin{aligned} L_1(\mathcal{M}) &= \mathcal{M}_*^{\text{op}} \\ L_\infty(\mathcal{M}) &= \mathcal{M}, \end{aligned}$$

where the operator space structure of  $\mathcal{M}_*^{\text{op}}$  is given by restriction of that of  $(\mathcal{M}_*^{\text{op}})^{**} = \mathcal{M}_{\text{op}}^*$ . Apart from being compatible with interpolation, such spaces satisfy that  $L_p(\mathcal{M})^* = L_{p'}(\mathcal{M}_{\text{op}})$ , see [Pis98] as well as the remarks on [Pis03, pp. 138]. It is also known that  $\mathcal{CB}(L_2(\mathcal{M})) = \mathcal{B}(L_2(\mathcal{M}))$ . The spaces  $L_p(\mathcal{M})$  can be turned into  $\mathcal{M}$ -bimodules. Indeed, let us have two commuting c.b. representations  $l_p : \mathcal{M} \rightarrow \mathcal{CB}(L_p(\mathcal{M}))$  and  $r_p : \mathcal{M}_{\text{op}} \rightarrow \mathcal{CB}(L_p(\mathcal{M}))$  generalizing the commuting actions in the GNS construction of  $\mathcal{M}$  when  $p = 2$ . The module structure of noncommutative- $L_p$  has been studied in [JS05]. Let us denote by  $S' \subset \mathcal{CB}(L_p(\mathcal{M}))$  the commutant of  $S \subset \mathcal{CB}(L_p(\mathcal{M}))$ , by [JS05, Theorem 1.5], we have that

$$\begin{aligned} l_p[\mathcal{M}]' &= r_p[\mathcal{M}_{\text{op}}] \\ r_p[\mathcal{M}_{\text{op}}]' &= l_p[\mathcal{M}]. \end{aligned}$$

Let us denote by  $\mathcal{CB}_{p,q}$  the operator space given by  $\mathcal{CB}(L_p(\mathcal{M}), L_q(\mathcal{M}))$ . Such spaces have a natural predual given by

$$\begin{aligned} \mathcal{CB}(L_p(\mathcal{M}), L_q(\mathcal{M})) &= \mathcal{CB}(L_p(\mathcal{M}), L_{q'}(\mathcal{M}_{\text{op}})^*) \\ &= \mathcal{CB}(L_p(\mathcal{M}), \mathbb{C}) \otimes_{\mathcal{F}} L_{q'}(\mathcal{M}_{\text{op}})^* \quad (\text{by [Pis03, Th. 4.1]}) \\ &= (L_p(\mathcal{M}) \hat{\otimes} L_{q'}(\mathcal{M}_{\text{op}}))^*. \end{aligned}$$

There are natural left actions on  $\mathcal{CB}_{p,q}$  by  $r_p[\mathcal{M}_{\text{op}}]$  and  $l_p[\mathcal{M}]$  and right actions by  $r_q[\mathcal{M}_{\text{op}}]$ ,  $l_q[\mathcal{M}]$ . We say that a subspace  $\mathcal{V} \subset \mathcal{CB}_{p,q}$  is a  $(p, q)$ -quantum relation over  $\mathcal{M}$  iff  $\mathcal{V}$  is weak-\* closed, with respect to the predual  $L_p(\mathcal{M}) \hat{\otimes} L_{q'}(\mathcal{M}_{\text{op}})$ , and a  $r_p[\mathcal{M}_{\text{op}}]$ - $r_q[\mathcal{M}_{\text{op}}]$ -bimodule. It is easily shown that such relations are independent of  $\phi$ . We also have the following.

**Proposition 4.1.**

- (i) *Quantum relations over  $\mathcal{M}$  correspond to  $(2, 2)$ -quantum relations.*
- (ii) *Projections in  $\mathcal{M} \overline{\otimes} \mathcal{M}_{\text{op}}$  are in bijective correspondence with  $(1, \infty)$ -quantum relations.*

**Proof.** The proof of (i) is immediate since  $\mathcal{CB}_{2,2} = \mathcal{B}(L_2(\mathcal{M}))$ ,  $l_2 : \mathcal{M} \rightarrow \mathcal{B}(L_2(\mathcal{M}))$  is the GNS representation of  $\mathcal{M}$  and  $r_2[\mathcal{M}]$  is just the commutant  $M'$  for that representation. In order to prove (ii) we need to use that the map  $j : \mathcal{M} \otimes_{\text{alg}} \mathcal{M} \rightarrow \mathcal{CB}(\mathcal{M}_*, \mathcal{M})$  given by linear extension of

$$j(x \otimes y)(\xi) = \langle y, \xi \rangle x$$

extends to a weak-\* continuous complete isomorphism  $j : \mathcal{M} \overline{\otimes} \mathcal{M} \rightarrow \mathcal{CB}(\mathcal{M}_*, \mathcal{M})$ , see [Pis03, Theorem 2.5.2]. we have that

$$\mathcal{CB}_{1,\infty} = \mathcal{CB}(L_1(\mathcal{M}), \mathcal{M}) = \mathcal{CB}(\mathcal{M}_*^{\text{op}}, \mathcal{M}) = \mathcal{M} \overline{\otimes} \mathcal{M}_{\text{op}}.$$

But, if  $T = j(x \otimes y)$ , we have that  $r_{\infty}(z) T r_1(t) = j(zx \otimes yt) = j((z \otimes t)(x \otimes y))$  and so a subspace  $\mathcal{V} \subset \mathcal{CB}_{1,\infty}$  is  $r_{\infty}[\mathcal{M}_{\text{op}}]$ - $r_1[\mathcal{M}_{\text{op}}]$ -bimodular iff, after seeing  $\mathcal{V}$  as a subspace of  $\mathcal{M} \overline{\otimes} \mathcal{M}_{\text{op}}$ , it is a left ideal. Since the map  $j$  is an isomorphism for the weak-\* topology,  $\mathcal{V} \subset \mathcal{M} \overline{\otimes} \mathcal{M}_{\text{op}}$  is also weak-\* closed. But any weak-\* closed left ideal is of the form  $\mathcal{V} = (\mathcal{M} \overline{\otimes} \mathcal{M}_{\text{op}}) P$ , where  $P \in \mathcal{P}(\mathcal{M} \overline{\otimes} \mathcal{M}_{\text{op}})$ .  $\square$

**Remark 4.1.** The result above explains intuitively why we cannot expect to define a well-behaved composition operation between projections  $P, Q \in \mathcal{P}(\mathcal{M} \overline{\otimes} \mathcal{M}_{\text{op}})$ . That composition will be carried to the composition of operators in  $\mathcal{CB}(L_1(\mathcal{M}), \mathcal{M})$  but that cannot be done, in general, since  $\mathcal{M}$  does not embeds canonically in  $L_1(\mathcal{M})$ .

It is natural to ask whether  $(p, q)$ -quantum relations are in correspondence with left ideals  $J \subset \mathcal{CB}_{r_q[\mathcal{M}_{\text{op}}]-r_p[\mathcal{M}_{\text{op}}]}^{\sigma}(\mathcal{CB}_{p,q})$  suitably closed in some weak topology. The following proposition asserts that this is the case when  $(p, q) = (1, \infty)$ .

**Proposition 4.2.** *The map  $\Phi^{1,\infty}(x \otimes y) = l_{\infty}(x) l_1(y)$  extends to a weakly continuous complete isomorphism*

$$\Phi^{1,\infty} : \mathcal{M} \overline{\otimes} \mathcal{M}_{\text{op}} \rightarrow \mathcal{CB}_{r_{\infty}[\mathcal{M}_{\text{op}}]-r_1[\mathcal{M}_{\text{op}}]}^{\sigma}(\mathcal{CB}_{1,\infty}).$$

*Under such correspondence any weakly closed left ideal  $J$  is of the form  $J = P(\mathcal{M} \overline{\otimes} \mathcal{M}_{\text{op}})$  and its associated bimodule  $V_J$  corresponds, under the bijection in (ii), to  $P^{\perp} \in \mathcal{P}(\mathcal{M} \overline{\otimes} \mathcal{M}_{\text{op}})$ .*

To prove the theorem above just notice that if  $\mathcal{N}$  is a von Neumann algebra, normal right  $\mathcal{N}$ -modular maps  $T : \mathcal{N} \rightarrow \mathcal{N}$  are given by left multiplication. Then, by applying that result to  $\mathcal{N} = \mathcal{M} \overline{\otimes} \mathcal{M}_{\text{op}}$  and using proposition 4.1 we conclude.

The discussion above leaves two natural open problems.

**Problem 4.2.**

- (P1.) Determine whether the double annihilator relation in Theorem 2.3 between modules and ideals holds in general for  $(p, q)$ -quantum relations.
- (P2.) define an operator space tensor product  $\otimes_{p,q}$  such that the map  $\Phi^{p,q} : \mathcal{M} \otimes_{\text{alg}} \mathcal{M}_{\text{op}} \rightarrow \mathcal{CB}_{r_q[\mathcal{M}_{\text{op}}]-r_p[\mathcal{M}_{\text{op}}]}^{\sigma}(\mathcal{CB}_{p,q})$  extends as a complete isometry to  $\mathcal{M} \otimes_{p,q} \mathcal{M}_{\text{op}}$ .

## 5. $W^*$ -metrics and c.b. Gaussian bounds

The aim of this section is to explain the original motivation that guided us into studying quantum relations. Such motivation was the necessity on [GPJP15] and [GPJP16] of expressing *off-diagonal* bounds in the context of noncommutative metric spaces. Recall that in the classical case an operator  $T = [a_{x,y}]_{x,y \in X}$  affiliated to  $\mathcal{B}(\ell_2 X)$  has off-diagonal bounds if certain norms of

$$[a_{x,y} \chi_{\{(x,y): d(x,y) > r\}}]_{x,y \in X}$$

decay in terms of  $r > 0$ . Earlier definition of *quantum metric spaces* in the  $C^*$ -algebraic framework, see [Rie04b], [Rie04a], do not provide a natural way of formulating such notion. On the other hand the notion of  $W^*$ -metric introduced by Kuperberg and Weaver in [KW12] seems particularly well suited to the task since a  $W^*$ -metric is a noncommutative generalization of the bundle of band matrices of width  $r > 0$ .

The upbringing of the notion of quantum relation is tightly connected with the concept of  $W^*$ -metric space introduced by Kuperberg and Weaver in [KW12]. Let us recall briefly such definition.

**Definition 5.1.** ([KW12, Definition 2.1(a)/2.3]) A family of subspaces  $\mathbb{V} = (\mathcal{V}_r)_{r \geq 0}$  of  $\mathcal{B}(H)$  is a  $W^*$ -pseudometric over  $\mathcal{M} \subset \mathcal{B}(H)$  iff

- (i) Each  $\mathcal{V}_r$  is a quantum relation over  $\mathcal{M}$ .
- (ii) Each  $\mathcal{V}_r$  is symmetric, i.e.  $\mathcal{V}_r^* = \mathcal{V}_r$ .
- (iii)  $\mathcal{V}_r \cdot \mathcal{V}_s \subset \mathcal{V}_{r+s}$ .
- (iv)  $\bigcap_{s > t} \mathcal{V}_s = \mathcal{V}_t$ .

We say that  $\mathbb{V}$  is a  $W^*$ -metric iff  $\mathcal{V}_0 = \mathcal{M}$ .

Notice that, if  $\mathcal{M} = \ell_{\infty}(X)$  is a discrete measure space, then every  $\mathcal{V}_r$  corresponds to a relation  $R_r \subset X \times X$ . Condition (ii) becomes usual symmetry for  $R_r$ . Defining a function  $d_{\mathbb{V}}(x, y) = \inf\{r : (x, y) \in R_r\}$  gives that (iv) is the triangular inequality and so  $d_{\mathbb{V}}$  is a classical (pseudo)metric.

Classically, a metric measure space is a triple  $(X, \mu, d)$  where  $\mu$  is a measure and  $d$  is a metric such that the Borel  $\sigma$ -algebra generated by  $d$  is composed of measurable sets. The noncommutative version of a measure space is generally regarded as a pair  $(\mathcal{M}, \tau)$ , where  $\mathcal{M}$  is a von Neumann algebra and  $\tau : \mathcal{M}_+ \rightarrow [0, \infty]$  normal, semifinite and faithful trace (or more generally a weight). Using  $W^*$ -metrics in this context gives a good noncommutative generalization of metric measure spaces. There are other, earlier, notions of quantized metric spaces, see for instance [Rie04b], but

$W^*$ -metrics have some advantages. One of them is that they provide a more natural framework for studying both finite speed of propagation and *off-diagonal* bounds associated with a *Markovian semigroup* over  $(\mathcal{M}, \tau)$ . Recall some definitions.

**Definition 5.2.** A semigroup  $(S_t)_{t \geq 0}$  of normal operators  $S_t : \mathcal{M} \rightarrow \mathcal{M}$  is said to be Markovian iff

- (i) Each  $S_t$  is unital and completely positive.
- (ii) The semigroup is symmetric, i.e.  $\tau((S_t x)^* y) = \tau(x^* S_t y)$ .
- (iii) The map  $t \mapsto S_t$  is pointwise weak-\*

Observe that as a consequence of  $S_t$  being unital and (ii) we get that  $\tau S_t = \tau$ .

The most classical example of such type of semigroup is given by the heat semigroup on  $\mathbb{R}^n$ . In such case  $S_t = e^{-t(-\Delta)}$  and its kernel  $k_t$  satisfies Gaussian bounds of the form

$$(5.1) \quad \|k_t(x, y) \chi_{\{(x, y) : d(x, y) > r\}}\|_{L_\infty(\mathbb{R}^n \times \mathbb{R}^n)} \lesssim_{(n)} \frac{e^{-\frac{r^2}{4t}}}{\sqrt{t^n}}.$$

Such bounds have been used in the noncommutative case in [GPJP15] with an ad hoc approach for  $\mathcal{M} = LG$ . Notice that, if  $J_{\mathcal{V}_r}$  is generated by a projection  $P_r \in \mathcal{P}(\mathcal{M} \overline{\otimes} \mathcal{M}_{op})$  and the semigroup  $S_t$  can be expressed as an integral operator by

$$S_t(x) = \tau\{k_t(\mathbf{1} \otimes x)\},$$

for some  $k_t$  affiliated to  $\mathcal{M} \overline{\otimes} \mathcal{M}_{op}$ , then the off-diagonal restriction is just  $k_t P_r$  and we can generalize (5.1) by bounding such element. Since, in general, ideals in  $\mathcal{M} \otimes_{eh} \mathcal{M}$  are not principal, such projection doesn't exist. Nevertheless, we can take  $\Phi_s(S_t)$  for  $s \in \text{Ball}(J_{\mathcal{V}_r})$ , the unit ball of  $J_{\mathcal{V}_r}$ , obtaining noncommutative Gaussian bounds of the form

$$\sup_{s \in \text{Ball}(J_{\mathcal{V}_r})} \|\Phi_s(S_t)\|_{CB(L_1(\mathcal{M}), \mathcal{M})} \lesssim \frac{e^{-\beta \frac{r^2}{t}}}{\sqrt{t^n}}.$$

Another Harmonic analysis concept that seems natural to formulate in the context of  $W^*$ -metrics is finite speed of propagation for the wave equation. Recall that if  $S_t = e^{-t(-\Delta)}$  is the heat equation in  $\mathbb{R}^n$  its associated wave equation is given by

$$\partial_t^2 f_t + (-\Delta) f_t = 0.$$

The solution of such equation have *finite speed of propagation*, meaning that if  $f_t$  is a solution of the above equation and  $\text{supp}[f_0] = K$ , after time  $t > 0$  the support of  $f_t$  is contained in

$$B_t(K) = \bigcup_{x \in K} B_t(x).$$

Such condition can be defined trivially using  $W^*$ -metrics as follows.

**Definition 5.3.** We say that a Markovian semigroup over  $S_t = e^{tA}$  have finite speed of propagation (with respect to some  $W^*$ -metric  $\mathbb{V}$ ) iff

$$\cos(t\sqrt{A}) \in \mathcal{V}_t, \quad \forall t > 0.$$

Observe that the definition makes perfect sense since, without loss of generality we can assume  $\mathcal{V} \subset \mathcal{B}(L_2(\mathcal{M}, \tau))$  and clearly  $\cos(t\sqrt{A})$  is bounded in  $L_2$ . The intuition behind is that  $x_t = \cos(t\sqrt{A})x$  satisfies the equation  $\partial_t^2 x_t + Ax_t = 0$  with  $x_0 = x$ .

Gaussian bounds and finite speed of propagation are equivalent after assuming certain hypothesis, see [Sik96] [Sik04]. Generalizing such results and exploring the connections with locality in the noncommutative setting is the goal of a forthcoming article.

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